COMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #34: Partisan Games
(from Conway, *On Numbers and Games*
and Berlekamp, Conway, and Guy, *Winning Ways*)
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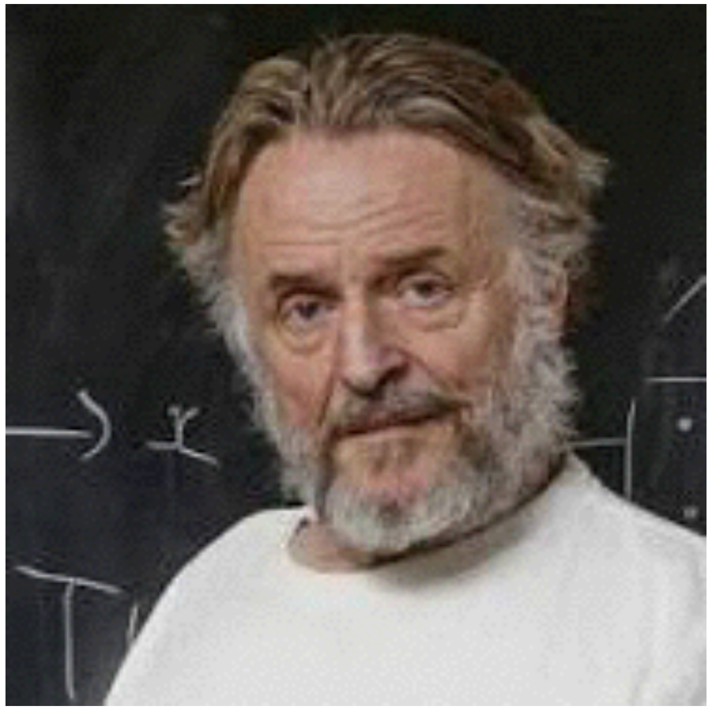
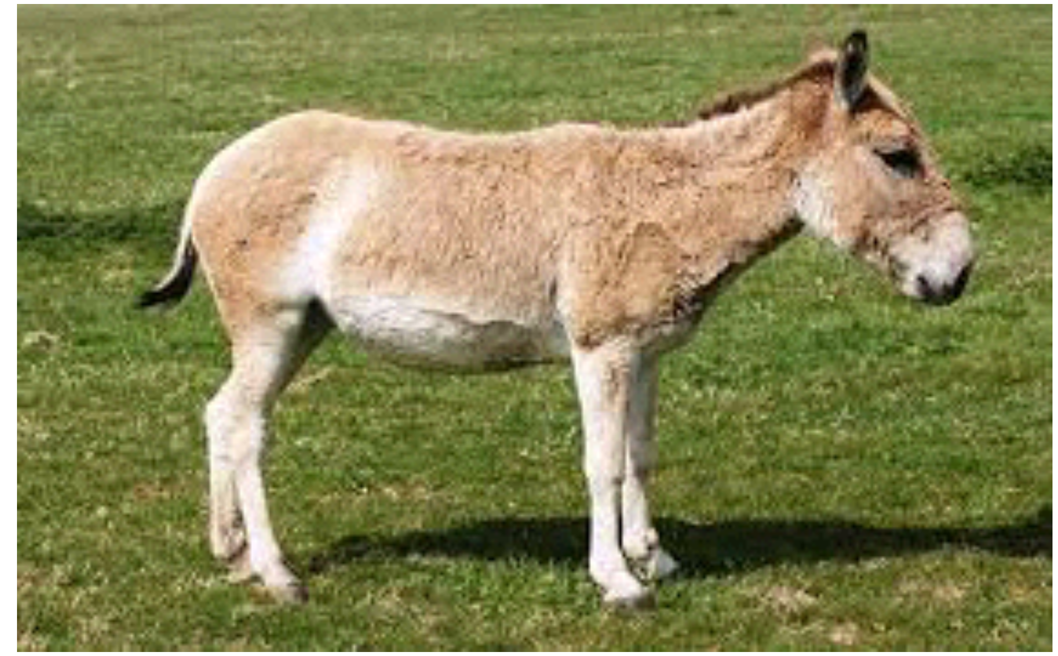
Partisan Games

- Conway's Game Theory
- Hackenbush and Domineering
- Four Types of Games and an Order
- Some Games are Numbers
- Values of Numbers
- Single-Stalk Hackenbush
- Some Domineering Examples

Conway's Game Theory





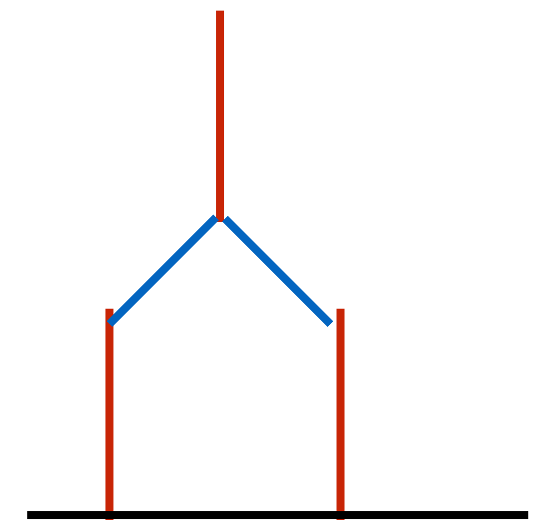
- **J. H. Conway** introduced his combinatorial game theory in his 1976 book *On Numbers and Games* or *ONAG*. Researchers in the area are sometimes called **onagers**.

- Another resource is the book *Winning Ways* by Berlekamp, Conway, and Guy.

Conway’s Game Theory

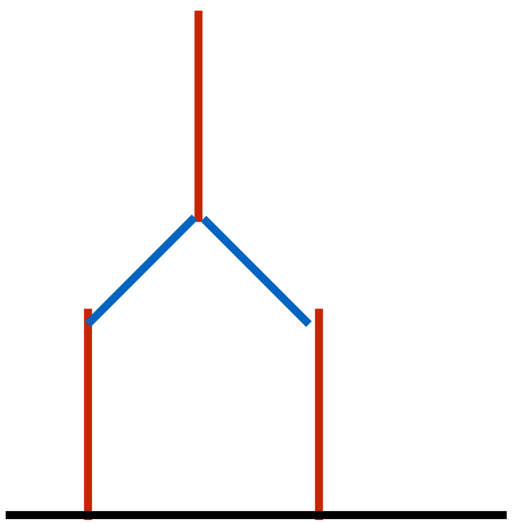
- Games, like everything else in the theory, are defined recursively. A **game** consists of a set of **left options**, each a game, and a set of **right options**, each a game.
- The base of the recursion is the **zero game**, with no options for either player.
- Last time we saw **non-partisan** games, where each player had the same options from each position. Today we look at **partisan** games.

Hackenbush

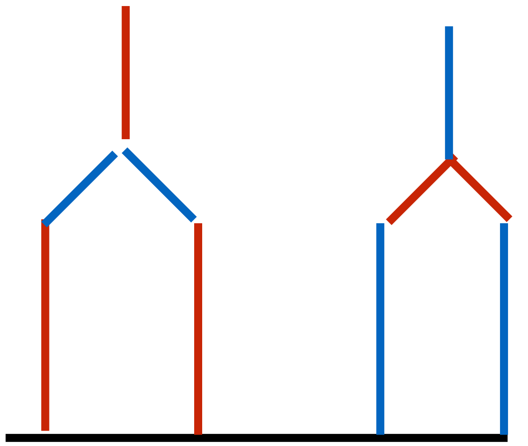


- **Hackenbush** is a game where the position is a diagram with red and blue edges, connected in at least one place to the "ground".
- A move is to delete an edge, a blue one for Left and a red one for Right.
- Edges disconnected from the ground disappear. As usual, a player who cannot move loses.

Hackenbush



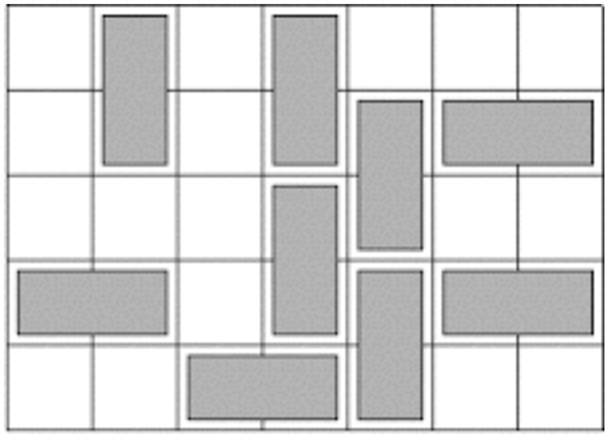
- From this first position, Right is going to win, because Left cannot prevent him from killing both the ground supports. It doesn't matter who moves first.



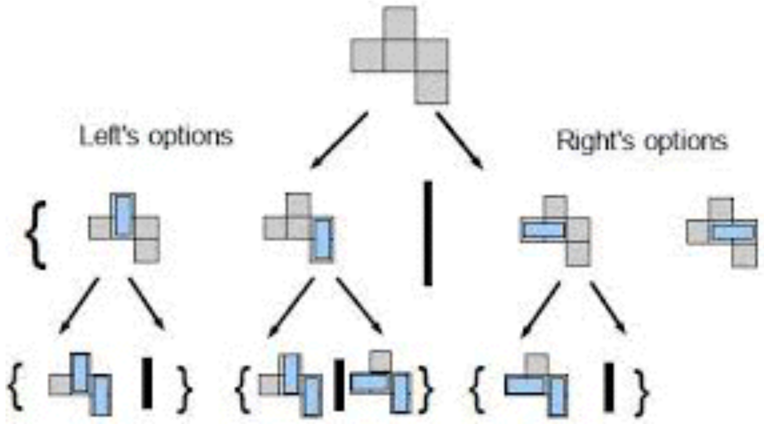
- In the second position, the first player to move is going to lose, whoever it is. This is a **zero game** according to our language about non-partisan games like Nim.

Domineering

- **Domineering** is a board game played on a square grid (or any set of squares on it) with 1 by 2 rectangles or **dominoes**.

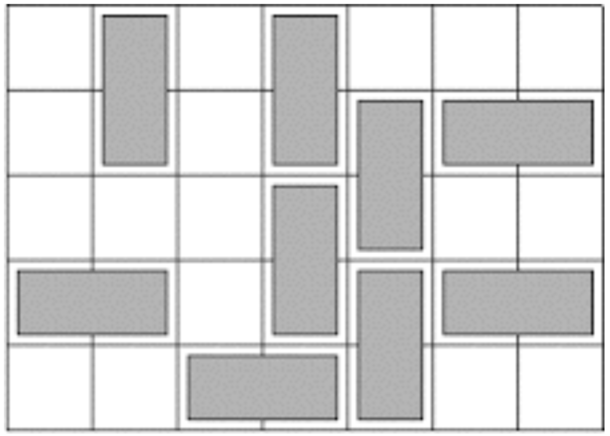


- A Left move is to place a vertical domino on two vacant squares; a Right move is to place a horizontal domino.

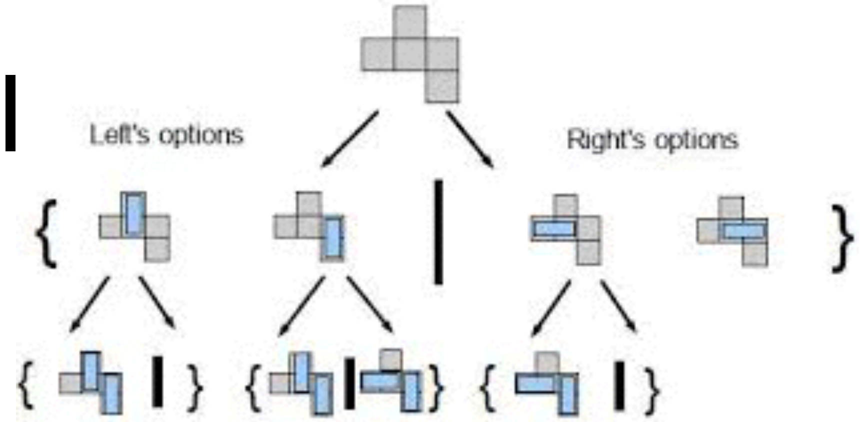


- Again, if you can't move you lose.

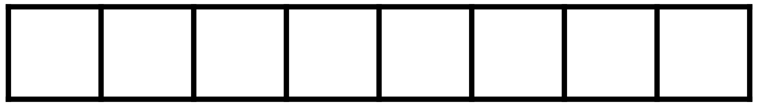
Domineering



- Of course an empty or 1 by 1 board is a zero game. An L-shaped three-square board is equivalent to a Nim pile of size 1.



- What about a 1 by n (horizontal) rectangle? Left cannot move at all but Right's moves produce smaller rectangles of the same type. He can make floor(n/2) free moves.



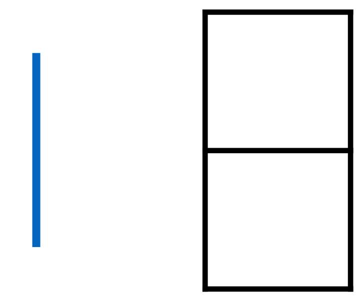
Four Types of Games

- We proved that every non-partisan game was either a first-player win or a second-player win, and we called the latter games **zero**. Conway's word for the former games is **fuzzy**.
- But now we have games that Left wins whether she starts or not, or that Right wins whether he starts or not. We call the first set **positive** and the second **negative**.

Four Types of Games

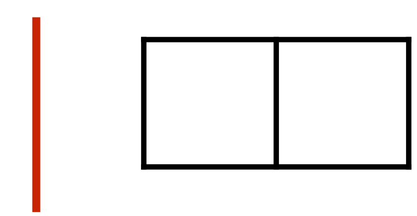
- The simplest positive game lets Left move to a zero game and gives right no moves. We'll call this game "+1".

+1



- The analogous game for Right is -1.

-1



- Under addition, these games act just like the integers +1 and -1, and we can make a game for any integer by summing them up.

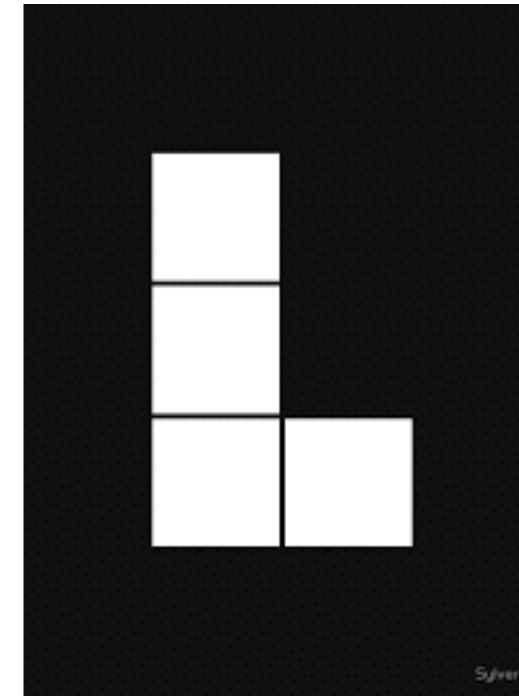
An Order For Games

- Any partisan game has an additive inverse, which we make by exchanging the roles of Left and Right. In Hackenbush we reverse the color of each edge, and in Domineering we reflect the board around a 45 degree line.
- By the Mirror Strategy, any game added to its inverse is a zero game.
- Given any two games G and H, we can form the game $G - H = G + (-H)$.

An Order For Games

- If G - H is a zero game, we say that G and H are **equal**.
- If G - H is a positive game, we say that G > H.
- If G - H is a negative game, we say that G < H.
- And if G - H is a fuzzy game, we say that G and H are **confused**.
- "G $\leq$ H" gives us a partial order on games because it is reflexive, antisymmetric, and transitive (prove these claims!).

Some Games are Numbers



- Now we are ready to define the subset of games that Conway calls **numbers**.
- A number is a game whose left and right options are all numbers, such that every Left option is smaller than every Right option.
- For example, in Domineering on the L tetromino, Left can move to 0 or -1, while Right can only move to 1.

Properties of Numbers

- If I add two numbers G and H, the result is also a number. To see this, look at the Left options in G + H: they are G_{L1}+H, G_{L2}+H,…, G_{Lk}+H, where the G_i's are the left options of G, and similarly G+H_{L1},…, G+H_{Lm}.
- Similarly we define the Right options of G+H.
- Since we always have $G_{Li} < G < G_{Rj}$ and similarly, and addition respects order, we can show that every Left option of G+H is less than every Right option.
- It then also follows that any two numbers are comparable under the order.

Single-Stalk Hackenbush

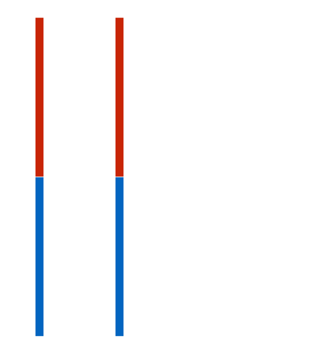
- We can easily find the value of a pattern where every stalk is all red or all blue, by subtracting the number of red edges from the number of blue. Things get interesting when the colors interact.

- Any single stalk with blue bottom is positive, since Left will always win. What's the value of this two-edge stalk?

Single-Stalk Hackenbush

- The value is less than 1, because if we add -1 we get a negative game.

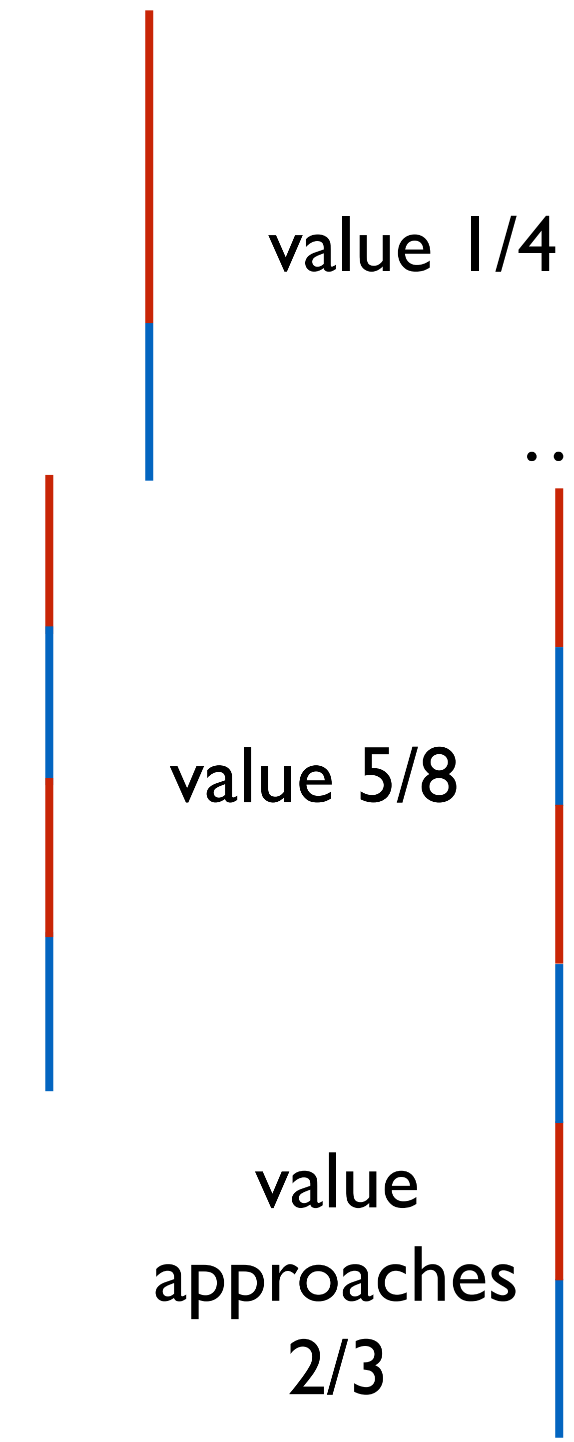
- But two of these stalks, plus -1, give a zero game (try it!), meaning that the value of each must be 1/2.



- What about other single stalks? The bottom edge determines the sign of the value. Adding a red edge to the top decreases the value, and adding a blue edge increases it.

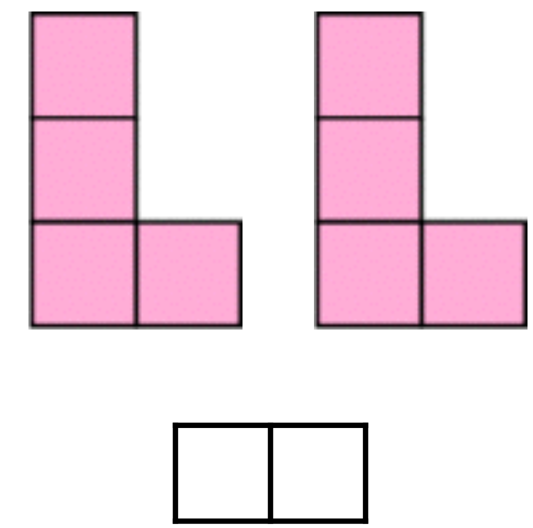
Single-Stalk Hackenbush

- With some work (or a more general theory) we can verify that the various finite single stalks have values that range over every rational number whose denominator is a power of two.
- This means that there is an infinite single stalk representing every real number, using its infinite binary expansion.

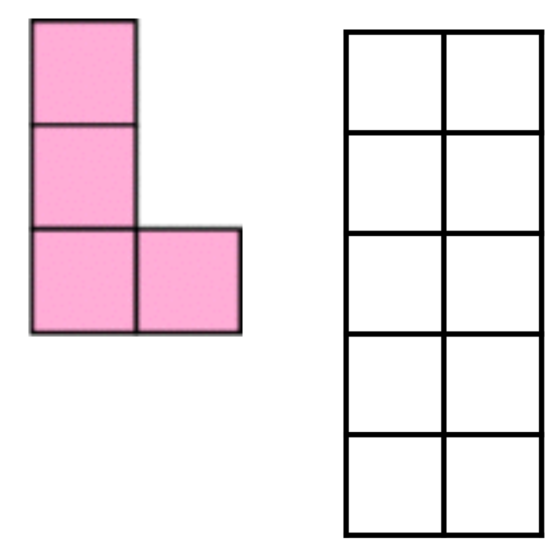


Some Domineering Examples

- We can show that the L tetromino has value 1/2, by adding two of them and -1 to get 0.



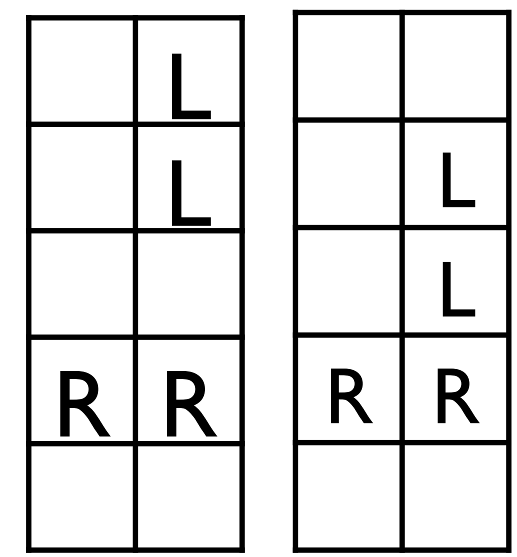
- The 5 by 2 rectangle actually has value -1/2, as we could show by analyzing the sum of it and the L.



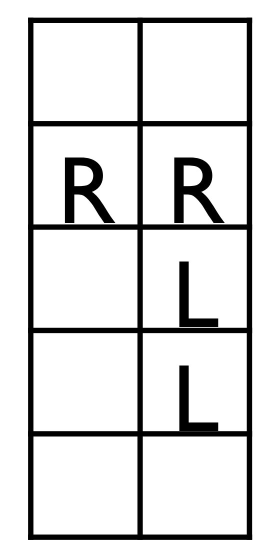
- But the 2 by 2 square, and in fact most Domineering positions, is neither a number nor a nimber.

Some Domineering Examples

- How do we show that the value of the 2 by 5 is -1/2?
- There are effectively only two first moves for Left, and in each Right can create a position of value 1/2 - 1 = -1/2.



- If Right moves first, he can leave a 1 by 2 and a 3 by 2, and any Left move creates 1/2 - 1 = -1/2.



- To finish we must show that Right's other moves are all worse than these.