Lecture #31: Polya’s Formula
(Tucker Section 9.4)
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Polya’s Formula

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Pattern Inventories

- We’ll finish our study of groups and symmetries by looking at pattern inventories for colorings.
- We’ve learned how to count equivalence classes of colorings defined by a group of symmetries.
- But the set of all colorings is already divided into subsets, based on the number of occurrences of each color.
Pattern Inventories

• With 2-colorings of the four vertices of a square, we could have 0, 1, 2, 3, or 4 white vertices, and no symmetry can alter that number. So there are at least five equivalence classes, and we saw already that there are 6.

• A more complete description than “six classes” is “one class with 0 whites, one with 1, two with 2, one with 3, and one with 4”.

• We can write this is as the polynomial $b^4 + b^3w + 2b^2w^2 + bw^3 + w^4$, the pattern inventory.
Two-Colored Squares Again

• Let’s see how we can calculate this pattern inventory in the case of 2-colored squares.

• The identity fixes all the colorings, and we can inventory all the colorings by the polynomial $b^4 + 4b^3w + 6b^2w^2 + 4bw^3 + w^4$. Note that this equals $(b+w)^4$.

• The two rotations fix only the mono-colored colorings, which have inventory $b^4 + w^4$. 
Two-Colored Squares Again

- The three double-flips each fix four colorings, inventoried by $b^4 + 2b^2w^2 + w^4 = (b^2+w^2)^2$.

- The two single flips each fix eight colorings, inventoried by $b^4 + 2b^3w + 2b^2w^2 + 2bw^3 + w^4 = (b+w)^2(b^2+w^2)$.

- Adding the eight inventories (one for each permutation in the group) gives us the inventory $8b^4 + 8b^3w + 16b^2w^2 + 8bw^3 + 8w^4$. Dividing this by 8 gives us the overall pattern inventory.
What’s Going On?

• We know that when we add up the elements fixed by each permutation, we get \(|G|\) copies from each equivalence class. This is why the cycle index polynomial, evaluated with \(r\) for each variable, gives us the number of classes.

• What’s happening now is that we are replacing each of those \(r\)'s by \(b\)'s and \(w\)'s, so that each monomial in the eventual sum becomes a monomial in \(b\)'s and \(w\)'s, marking the number of uses of each color in that coloring.
What’s Going On?

- The identity permutation, for example, has four 1-cycles and fixes all $2^4$ colorings. The polynomial $(b+w)^4$ has one monomial for each of these colorings.

- A double-flip, by contrast, has two 2-cycles, and a color is assigned to each 2-cycle in a coloring fixed by it. Since each cycle has two blacks or two whites, the polynomial $(b^2+w^2)^2$ inventories those fixed colorings.
Polya’s Formula

• Recall that the cycle index polynomial $P_G$ for a group $G$ is $1/|G|$ times the sum, for each element $\pi$ of $G$, of a monomial giving the cycle structure of $\pi$.

• Polya’s theorem says that if we substitute $b + w$ for $x_1$, $b^2 + w^2$ for $x_2$, and similarly $b^k + w^k$ for each $x_k$, and evaluate $P_G$ with those values, we get the pattern inventory for 2-colorings.

• With more than two colors we use the sum of the $k^{th}$ powers of a variable for each color.
Rotations of a Triangle

• Let’s look at this with $G$ as $\mathbb{Z}_3$ and $S$ as a triangle, so that $G$ is the group of rotations.

• The cycle index polynomial is $(x_1^3+2x_3)/3$, and substituting we get a pattern inventory of $((b+w)^3+2(b^3+w^3))/3 = b^3+b^2w+bw^2+w^3$. This represents the four classes of 2-colorings.

• With three colors we get $((b+w+r)^3+2(b^3+w^3+r^3))/3 = b^3+w^3+r^3+b^2w+b^2r+w^2b+w^2r+r^2b+r^2w+2bwr$. The number of colors determines the class except for $bwr$. 
Rotations of a Heptagon

• With a 7-gon the cycle index polynomial for rotations is \( (x_1^7+6x_7)/7 \), so Polya’s formula for 2-colorings gives us \( ((b+w)^7+6(b^7+w^7))/7 \).

• The series of coefficients for \( (b+w)^7 \) is a line of Pascal’s Triangle, \( (1 \ 7 \ 21 \ 35 \ 35 \ 21 \ 7 \ 1) \). The other term has coefficients \( (6 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 6) \), so the sum is \( (7 \ 7 \ 21 \ 35 \ 35 \ 21 \ 7 \ 7) \). Dividing by 7 and reverting to polynomial notation gives \( b^7+b^6w+3b^5w^2+5b^4w^3+5b^3w^4+3b^2w^5+bw^6+w^7 \). There are 20 total classes and we have the pattern inventory.
Edges of a Tetrahedron

• The group $A_4$ of symmetries of a tetrahedron also acts on the six edges of the tetrahedron. The cycle index polynomial for that action is $(x_1^6 + 8x_3^2 + 3x_1^2x_2^2)/12$, as we can see by analyzing the eight 120-degree rotations about a point and the three double-flips.

• Substituting $(b^k + w^k)$ for $x_k$ gives us the sum of three polynomials with coefficients $(1 6 15 20 15 6 1)$, $(8 0 0 16 0 0 8)$, and $(3 6 9 12 9 6 3)$. The sum is $(12 12 24 48 24 12 12)$ and the pattern inventory is $b^6 + b^5w + 2b^4w^2 + 4b^3w^3 + 2b^2w^4 + bw^5 + w^6$. 
Vertices of a Cube

• One more example is the symmetries of a cube. There are 24, because we could have any of the six sides on the bottom in any of four orientations. The group is isomorphic to $S_4$, but is acting on the eight vertices.

• The cycle index polynomial takes some work to compute: $(x_1^8 + 6x_4^2 + 9x_2^4 + 8x_1^2x_3^2)/24$.

• We add $(1 8 28 56 70 56 28 8 1), (6 0 0 0 12 0 0 0 6), (9 0 36 0 54 0 36 0 9), and (8 16 8 16 16 32 16 8 16 8)$, divide by 24, and get $b^8 + b^7w + 3b^2w^6 + 3b^3w^5 + 7b^4w^4 + 3b^3w^5 + 3b^2w^6 + bw^7 + w^8$. 
Undirected Graphs

- Consider the set of possible edges in an n-vertex undirected graph. A permutation of the vertices also permutes the edges. So we can think of the group $S_n$ as acting on the edges, with a cycle index polynomial.

- A particular undirected graph can be thought of as a two-coloring of the edges, with one color for “edge” and one for “non-edge”. (This is the basis of Exercise 9.4.16 on HW#7.)
Undirected Graphs

• Two n-vertex graphs are isomorphic if there is a permutation of the vertices that takes one to the other. Thus the number of graphs, up to isomorphism, is the number of 2-colorings of the edge set, up to the action of $S_n$ on that edge set, and can be computed by the methods we’ve used here.

• Polya’s Formula can give us a pattern inventory of these “colorings”, which is an inventory of the graphs by number of edges.
Undirected Graphs

- In 2007 I wanted a list of all the graphs of various small sizes, up to isomorphism. I also wanted a list of “two-colored graphs”, which correspond to 3-colorings of the edge set with “no edge”, “red edge”, and “blue edge”.

- I wrote a computer program to generate these lists, and the results are on my web site. I just made a backtrack search through the possibilities, rejecting any graph that was isomorphic to a lexicographically smaller graph.
Undirected Graphs

• Once you have solved Exercise 9.4.16, you will know how to use Polya’s Formula to get pattern inventories for each of these lists of graphs.

• Of course to construct the inventories you would need the cycle index polynomial of the action of $S_n$ on the edges, for each $n$.

• For $S_3$ the action on the three edges is the same as that on the three vertices. But for $S_4$ things already get more interesting.