COMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #29: Burnside's Theorem
(Tucker Section 9.2)
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Burnside's Theorem

- Group Actions on Colorings
- Equal Equivalence Classes
- Colorings of a One-Sided Triangle
- Two Forms of Burnside’s Theorem
- Example: Two-Banded Batons
- Example: Three-Banded Batons

Group Actions on Colorings

- We're currently engaged in counting colorings of a figure with **symmetries**, our main example being a two-sided square with colored vertices.
- The symmetries of a two-sided square are eight, four rotations by multiples of 90 degrees and four transformations that are the composition of a reflection and one of these rotations.
- These symmetries form a **group** under the operation of functional composition.

Group Actions on Colorings

- In general, if we have a set S of objects, and a group G of bijections from S to S, we can ask how many r-colorings of S there are, when two colorings are considered to be the same when an element of G takes one to another.
- Each element of G induces an **action** on the colorings of S. We have an equivalence relation on colorings, where a coloring c is equivalent to every member of its **orbit** under G, the set $\{f(c): f \in G\}$.

Equal Equivalence Classes

- The number of colorings up to symmetry is the number of orbits, or the number of equivalence classes.
- In some cases the orbits all have the same size, making them easy to count.
- For example, consider colorings of the vertices of an r-gon using each of r colors exactly once. G here will be just the set of rotations, without reflections, a group of r elements (called $\mathbb{Z}_r$, the integers mod r).

Equal Equivalence Classes

- There are r! colorings of the r-gon with exactly r colors. If c is any coloring, the orbit of c consists of exactly r colorings.
- The number of equivalence classes is thus just $r!/r = (r-1)!$.
- What if we add rotations? Now each orbit has exactly 2r colorings in it (assuming $r > 2$) and the number of classes is $(r-1)!/2$.
- But things are more complicated for general colorings, where objects have the same color.

Coloring One-Sided Triangles

- Let's look at the eight 2-colorings of a triangle, with G being the group $\mathbb{Z}_3$ of three rotations.
- We can't just divide the number of colorings (8) by |G|, as this isn't even an integer.
- There are in fact four classes, as any two colorings with the same number of white vertices are equivalent. Two classes (0 and 3 whites) have one member, the other two (1 and 2 whites) have three each.

Coloring One-Sided Triangles

- If we look at 3-colorings of triangles under rotation, we find that the 27 colorings divide into eleven classes, three of size 1 (BBB, RRR, WWW) and eight of size 3. (Note that BWR and BRW are different with just rotations.)
- Under the group of six symmetries of a two-sided triangle, with both rotations and reflections, we get ten classes, three of size 1, six of size 3, and one of size 6.

Burnside's Theorem

- Burnside's Theorem allows us to count the number N of equivalence classes of objects under a group action.
- Let T be any collection of colorings of S that is closed under G. For any x in T, let $\varphi(x)$ be the number of elements of G that leave x fixed. Then N is $1/|G|$ times the sum, for all x in T, of $\varphi(x)$.
- Why is this true?

Burnside's Theorem

- Consider the orbit of any element x of T. If we count the elements of this orbit with multiplicities, we will get exactly $|G|$. Every element f of G takes x somewhere, and the number that take it to each element of the orbit is exactly $\varphi(x)$.
- Counting $\varphi(x)$ for each element of the orbit, we get $|G|$ total, which means that each orbit counts 1 in the sum after we divide by $|G|$.

Alternate Burnside's Theorem

- We can think of this same sum in another way. For every element π of G, let $\Psi(\pi)$ be the number of colorings in T that are fixed by π. Then N = $1/|G|$ times the sum, over all π in G, of $\Psi(\pi)$.
- This is true because if π fixes x, we count it exactly once in the sum of the $\varphi(x)$'s and exactly once in the sum of the $\Psi(\pi)$'s. So the two sums are exactly the same.

Example: Two-Banded Batons

- Let's look at a simple example where S is the set of two ends of a baton, and our group G of symmetries has just two elements: the identity function and the function that exchanges the two ends of the baton.
- How many r-colorings of this baton are possible, up to symmetry?
- There are r^2 colorings before we take symmetry into account.

Two-Banded Batons

- If $r = 2$, there are four possible batons, BB, BW, WB, and WW. The identity permutation fixes all four, but the flip fixes only BB and WW.
- Since $\Psi(1) = 4$ and $\Psi(\text{flip}) = 2$, the sum of $\Psi(\pi)$ is 6 and $N = (1/|G|)6 = 3$.
- For 3-colorings, the identity fixes all nine batons, and the flip only the three that are a single color. So $N = (1/2)(9+3) = 6$.
- In general for r-colorings $N = (1/2)(r^2+r)$.

Two-Banded Batons

- We did these calculations using the alternate form of Burnside's Theorem. What about the original form?
- Of the r^2 colorings, the r single-color ones have $\varphi(x) = 2$ because both the identity and the flip fix them. The other r^2-r colorings have $\varphi(x) = 1$ because only the identity fixes them.
- So $N = (1/2)(r(2) + (r^2-r)(1)) = (r^2+r)/2$.

Three-Banded Batons

- Now consider three-banded batons with the same two-element group G. There are now r^3 colorings to start.
- To determine $\varphi(x)$ for some x, we need to know whether the flip fixes x. This is true if and only if the colors on the two ends are the same. So r^2 x's have $\varphi(x) = 2$ and the other r^3-r^2 have $\varphi(x) = 1$.
- Thus $N = (1/2)(r^2(2) + (r^3-r^2)) = (r^3+r^2)/2$.

Three-Banded Batons

- Alternatively, we can determine N by seeing how many colorings are fixed by each element of G.
- The identity fixes all r^3, and the flip fixes r^2, so we have $N = (1/2)(r^3+r^2)$.
- What about four bands? Now the identity fixes all r^4 and the flip fixes r^2, so $N = (r^4+r^2)/2$.
- In general, for 2k bands we get $N = (r^{2k}+r^k)/2$ and for 2k+1 bands we get $N = (r^{2k+1}+r^{k+1})/2$.