Lecture #24: Recurrences: D&C and Linear
(Tucker Sections 7.2, 7.3, and 7.4)
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Recurrences

• Systems of Recurrences
• Divide and Conquer Recurrences
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• Linear Recurrences
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• Inhomogeneous Recurrences
• Compound Inhomogeneous Terms
I am in an Opera!

- Valley Light Opera is doing Gilbert and Sullivan’s *Ruddigore* November 12, 13, 18, 19, and 20 at the Academy of Music in Noho.

- Preview show Friday 11th at 7:30 costs $5 instead of $15 or $10 college rush at other shows: see vlo.org.
Systems of Recurrences

• Sometimes we need more than one recurrence to solve a counting problem.

• Consider strings over \{a,b,c\} with an even number of b’s and an odd number of c’s.

• If \( f(n) \) is the number of such strings of length \( n \), we have that \( f(n) = f(n-1) + g(n-1) + h(n-1) \), where \( g(n) \) is the number with odd numbers of both b’s and c’s, and \( h(n) \) the number with even numbers of each.
Systems of Recurrences

• Then $g(n)$, for example, is $g(n-1) + f(n-1) + i(n-1)$, where $i(n)$ is the number with an even number of b’s and an odd number of c’s.

• Each of the four functions is defined by a recurrence using itself and two of the others.

• By induction on $n$, assuming we define $f(0)$, $g(0)$, $h(0)$, and $i(0)$ to each be 1, we have well-defined and correct values $f(n)$, $g(n)$, $h(n)$, and $i(n)$ for each $n$. 
Many algorithms take a divide and conquer approach, reducing a problem to similar problems with smaller parameters. Much of COMPSCI 311 is spent analyzing the resources used by such algorithms, and recurrences are a key tool in this analysis.

If $a_n$ is the number of steps to solve a problem of size $n$, we often get a recurrence of the form $a_n = ca_{n/2} + f(n)$, where $c$ is a constant and $f(n)$ is the time to split and merge the subproblems.
Simple D&C Examples

• If $c = 1$ and $f(n)$ is constant, we have $a_n = a_{n/2} + d$, which solves to $a_n = d\log_2(n) + A$, where $A$ is a constant chosen to fit the initial conditions. We assume here that $n$ is a power of 2, to avoid ceilings and floors.

• If $c = 2$ and $f(n)$ is constant, we have $a_n = 2a_{n/2} + d$, which solves to $a_n = An - d$. Our $3n/2 - 2$ steps to find max and min fits into this case.

• If $c = 2$ and $f(n) = dn$, we have $a_n = 2a_{n/2} + dn$, which solves to $a_n = dn(\log_2n + A)$. 
Fast Multiplication

• Normally multiplying two n-bit numbers would require $O(n^2)$ bit multiplications.

• By adding some cheaper additions, though, we can do it with fewer multiplications.

• Write the numbers $w_1$ and $w_2$ as $u_1v_1$ and $u_2v_2$, where the u’s and v’s are $n/2$ bit numbers. Then $w_1 \times w_2 = (u_1 \times u_2)2^n + [(u_1 \times v_2) + (v_1 \times u_2)]2^{n/2} + (v_1 \times v_2)$. We have four products of $n/2$-bit numbers.
Fast Multiplication

• \((u_1 \times u_2)2^n + [(u_1 \times v_2) + (v_1 \times u_2)]2^{n/2} + (v_1 \times v_2)\) has four products of \(n/2\)-bit numbers.

• But if we compute \(u_1 \times u_2, v_1 \times v_2,\) and \((u_1 + v_1) \times (u_2 + v_2),\) using only three multiplications, we can get all three terms we need by addition.

• Our number of multiplications satisfies the recurrence \(a_n = 3a_{n/2},\) which turns out to solve to \(a_n = n^{\log_3 3} = n^{1.585...},\) much better than \(n^2.\) Of course there are complications like the time for the additions.
The CLRS Master Theorem

• In COMPSCI 311 we learn a theorem called the Master Theorem in the popular CLRS textbook. It gives a solution to the recurrence $a_n = ca_{n/k} + f(n)$, which applies when we divide the size-$n$ problem into $c$ problems of size $n/k$ each, with $f(n)$ overhead to split the problems and merge the solutions.

• The solutions are given in big-O form, befitting a course where we usually regard resource bounds this way.
The CLRS Master Theorem

- We have $a_n = ca_{n/k} + f(n)$.
- The result depends on the relationship between $f(n)$ and $g(n) = n^{\log_c k}$, where the log is base $k$. The statement below is approximate.
- If $f(n) = o(g(n))$, then $a_n = \Theta(g(n))$.
- If $f(n) = \Theta(g(n))$, then $a_n = \Theta(g(n)\log n)$.
- If $f(n) = \omega(g(n))$, then $a_n = \Theta(f(n))$. 
Linear Recurrences

• A linear recurrence is one where the new term \( a_n \) is given by a linear combination of the \( r \) most recent terms, by a rule of the form \( a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ra_{n-r} \).

• Since \( a_k \) is not defined for negative \( k \), we have to give initial conditions \( a_1, \ldots, a_{r-1} \) as well as the usual \( a_0 \).

• There’s a general solution for these, which is reminiscent of the general solution for linear differential equations.
Solving Linear Recurrences

• It turns out that every such equation has a set of solutions that are themselves linear combinations of sequences of the form $\alpha^n$, for some fixed numbers $\alpha$.

• If $\alpha$ is going to lead to such a solution, we need to have $\alpha^n = c_1\alpha^{n-1} + \ldots + c_r\alpha^{n-r}$, which we can reduce to $\alpha^r = c_1\alpha^{r-1} + \ldots + c_r$, by dividing the first equation by $\alpha^{n-r}$. 
Solving Linear Recurrences

• So $\alpha$ must satisfy the equation $\alpha^r - c_1 \alpha^{r-1} - c_2 \alpha^{n-2} - \ldots - c_r = 0$, which is called the characteristic equation of the recurrence.

• Over the complex numbers, at least, this equation of degree $r$ has exactly $r$ roots, counting multiplicity. (We’ll assume for the time being that all the roots are distinct.)

• If $\alpha_1, \ldots, \alpha_r$ are the roots, any function of the form $A_1 \alpha_1^n + \ldots + A_r \alpha_r^n$ will be a solution to the recurrence, and these are all of them.
Dealing With Initial Conditions

• More precisely, any linear combination of the functions $\alpha_i^n$ will satisfy the rule of the recurrence. In order to satisfy the initial conditions as well, we need to set the $A_i$’s.

• If $a_0’,…, a_{r-1}’$ are the values of $a_0,…, a_{r-1}$ given by the initial conditions, then for every $k$ with $0 \leq k \leq r-1$, we must have $A_1\alpha_1^k + A_2\alpha_2^k + \ldots + A_r\alpha_r^k = a_k’$.

• These are $r$ equations in $r$ unknowns, and have exactly one solution.
Example: Exponential Rabbits

- If we start at time 0 with six rabbits, and the population doubles each year, how many do we have after $n$ years?

- The recurrence is $a_n = 2a_{n-1}$, with initial condition $a_0 = 6$. (Since $r = 1$ here, we need only one initial condition.) The characteristic equation is $\alpha^1 - 2 = 0$, solving to $\alpha = 2$.

- So any function of the form $A2^n$ meets the rule, and to have $A2^0 = 6$, we choose $A = 6$. 
Example: Fibonacci Rabbits

• The Fibonacci function was also originally designed to model rabbit populations, with each rabbit producing one offspring in every generation except its first. We don’t model any rabbit deaths.

• So the population $a_n$ after $n$ generations is the $a_{n-1}$ from the previous generation, plus one more for each of the $a_{n-2}$ rabbits that are more than one generation old.
Example: Fibonacci Rabbits

- So the characteristic equation is $\alpha^2 - \alpha - 1 = 0$, which by the quadratic formula has two roots, $\alpha_1 = (1+\sqrt{5})/2$ and $\alpha_2 = (1-\sqrt{5})/2$.

- Any function of the form $A_1\alpha_1^n + A_2\alpha_2^n$ follows the recursive rule. Solving the pair of equations $A_1 + A_2 = a_0' = 1$ and $A_1\alpha_1 + A_2\alpha_2 = a_1' = 1$ gives us $A_1 = \alpha_1/\sqrt{5}$ and $A_2 = -\alpha_2/\sqrt{5}$.

- It’s perhaps surprising that these irrational coefficients and bases of powers give us the familiar sequence 1, 1, 2, 3, 5, 8, 13, 21,…
Complex Roots

• How would we get complex numbers in the solution to a linear recurrence? Let $a_n = -a_{n-1} - a_{n-2}$, with initial conditions $a_0 = 0$ and $a_1 = 1$.

• We get a sequence 0, 1, -1, 0, 1, -1, 0, 1, -1,…, which doesn’t look exponential at all.

• But in fact the characteristic equation $\alpha^2 + \alpha + 1 = 0$ has two roots $(-1 + \sqrt{-3})/2$, the two complex cube roots of unity. The correct linear combination of powers of these gives the real numbers of our periodic sequence.
Multiple Roots

• What about a rule like $a_n = 4a_{n-1} - 4a_{n-2}$, with characteristic equation $\alpha^2 - 4\alpha + 4 = 0$, which has a double root of $\alpha = 2$?

• If we start with $a_0 = 0$ and $a_1 = 1$, the sequence goes on 4, 12, 32, 80, 192, which we might recognize as $a_n = n2^{n-1}$. Where did this come from?

• The function $2^n$ satisfies the rule for the recurrence, but it turns out that $n2^n$ does as well, as $n2^n = 4(n-1)2^{n-1} - 4(n-2)2^{n-2}$. 
Multiple Roots

• If $\alpha$ is a root of the characteristic equation with multiplicity $m$, then it turns out that $n\alpha^n$, $n^2\alpha^n$, ..., $n^{m-1}\alpha^n$ all satisfy the rule, and any function that satisfies the rule is a linear combination of these functions and $\alpha^n$ itself.

• We won’t prove this here, but on HW#6 you’ll show the $m = 3$ case (Exercise 7.3.9, not in the back of the book).

• Note that for a characteristic equation of degree $r$, we still have exactly $r$ functions, so that there will be one linear combination meeting the initial conditions.
Inhomogeneous Linears

• For example, consider a recurrence of degree 1, so that $a_n = ca_{n-1} + f(n)$. The h-part is “$ca_{n-1}$” and the i-part is $f(n)$.

• If we can find any function $z$ such that $z_n = cz_{n-1} + f(n)$, then any function of the form $a_n = Ac^n + z_n$ will satisfy the recurrence, and we can use the initial conditions to find $A$ as before.

• A special case is when $c = 1$, so that $a_n = a_{n-1} + f(n)$. This has the solution $z_n = f(1) + \ldots + f(n)$, so that $a_n = a_0 + f(1) + f(2) + \ldots + f(n)$.
Inhomogeneous Linears

- To solve $a_n = ca_{n-1} + f(n)$ with $c \neq 1$, we can use some standard general solutions, that we will justify with generating functions next lecture.

- If $f(n)$ is a constant $d$, the particular solution is another constant $B$.

- If $f(n) = dn$, we have $B_1n + B_0$ for two constants $B_0$ and $B_1$.

- For $f(n) = dn^2$ we have $B_2n^2 + B_1n + B_0$, and for $f(n) = ed^n$ we have $Bd^n$. 
Compound Inhomogeneous

- Example 3 in Section 7.4 has the recurrence \( a_n = 3a_{n-1} - 4n + 3 \cdot 2^n \), and asks for a general solution.

- The difficulty here is that the i-part is the sum of two functions, but we can proceed by finding particular solutions for each of the two functions, and the adding them to \( A3^n \).

- To get \( y_n = 3y_{n-1} - 4n \), we look for a solution of the form \( B_1n + B_0 \), and get \( 2n + 3 \) by taking the equation \( B_1n + B_0 = 3(B_1n + B_0) - 4n \) and solving for \( B_0 \) and \( B_1 \).
Example 3 in Section 7.4 has the recurrence \( a_n = 3a_{n-1} - 4n + 3 \cdot 2^n \), and asks for a general solution.

To get \( y_n = 3y_{n-1} + 3(2^n) \), we look for a solution of the form \( B2^n \), and get \( 6(2^n) \) by taking the equation \( B2^n = 3B2^{n-1} + 3(2^n) \) and solving for \( B \).

Our general solution is \( a_n = A3^n + 2n + 3 + 6(2^n) \).