CMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #20: Calculating Coefficients of GF's
(Tucker Section 6.2)
David Mix Barrington
28 October 2016

Calculating Coefficients of GF's

- Multiplying Power Series
- Division, and Two Identities
- Binomial Theorem Identities
- Powers of 1/(1-x)
- Finding One Coefficient
- Some Easy Counting Problems
- A Harder Counting Problem
- Verifying an Identity

Multiplying Power Series

- As we saw last time, we can take two power series $a_0+a_1x+a_2x^2+\ldots$ and $b_0+b_1x+b_2x^2+\ldots$ and get a product $c_0+c_1x+c_2x^2+\ldots$, by the rule that c_n is the sum for i from 0 to n of a_ib_{n-i}.
- This means that any *finite* product of power series is defined as a power series. If the factors have integer coefficients, so does the product.
- When can we *divide* power series?

Dividing Power Series

- If f, g, and h are power series and we know that fg = h, it's legitimate to say that g = h/f or f = h/g.
- In many cases we can take h and f and compute h/f by **long division**. (This may or may not keep us within integer coefficients.)
- In particular, suppose h is the GF for the sequence a_i and f the GF for b_i. We can compute h/f if $b_0 = 1$.

Dividing Power Series

- For example, let $a_i = 1$ for all i and let $f = 1+2x$. We can subtract f from h to get the series $-x+x^2+x^3+\ldots$, then subtract $-xf$ to get $3x^2+x^3+x^4+\ldots$, and so on.
- Carrying out this process gives us a quotient of $1-x+3x^2-5x^3+11x^4-21x^5+43x^6-\ldots$, and multiplying back together verifies this fact. But we'd rather have an analytic answer than just a sequence of coefficients.

Two Division Identities

- It turns out that dividing by $1-x$ gives us particular power series that are useful in combinatorics.
- It's easy to verify that $(1-x)(1+x+x^2+\ldots) = 1$, which means that $1/(1-x)$ is $(1+x+x^2+\ldots)$, the GF for the sequence with $a_r = 1$ for all r.
- And $(1-x^{k+1})/(1-x)$ is also easily verified to be $1+x+x^2+\ldots+x^k$, a series that came up several times last lecture.

Binomial Theorem Identities

- We also noted a number of identities that come from the Binomial Theorem.
- $(1+x)^n = C(n, 0) + C(n, 1)x + \ldots + C(n, n)x^n$.
- $(1-x)^n = C(n, 0) - C(n, 1)x + C(n, 2)x^2 - \ldots + (-1)^n C(n, n)x^n$.
- $(1-x^m)^n = C(n, 0) - C(n, 1)x^m + C(n, 2)x^{2m} - C(n, 3)x^{3m} + \ldots + (-1)^n C(n, n)x^{nm}$.

Powers of $1/(1-x)$

- Let's now look at what happens when we multiply $1/(1-x)$ by itself.
- $(1+x+x^2+\ldots)(1+x+x^2+\ldots)$ has one term of degree 0, two of degree 1, three of degree 2, and so forth, so that $1/(1-x)^2 = 1+2x +3x^2+4x^3+\ldots$
- What about $1/(1-x)^k$? We get a term of degree r for every possible sum $e_1+\ldots+e_k = r$, and we know there are $C(r+k-1, r)$ of those.

Powers of 1/(1-x)

- With k=2, C(r+1-1, r) is just r+1, so as we just saw, $1/(1-x)^2 = 1+2x+3x^2+4x^3+\ldots$
- With k=3, we have C(r+2, r) as our general term, for $1+3x+6x^2+10x^3+\ldots$
- With k=4 we have C(r+3, 3) which gives us $1+4x+10x^2+20x^3+35x^4+\ldots$
- The point is that we can easily get any coefficient of any power of 1/(1-x).

Finding One Coefficient

- Let's now start in on Tucker's examples of coefficient calculations, applying the tools we have built up.
- What is the coefficient of x^{16} in the series $(x^2+x^3+x^4+x^5+\ldots)^5$? We rewrite the series in parentheses as $x^2/(1-x)$, so our series is $x^{10}/(1-x)^5$. Thus we want the coefficient of x^6 in $(1-x)^5$, which is just $C(6+5-1, 6) = C(10, 4) = 10\times9\times8\times7/1\times2\times3\times4 = 10\times3\times7 = 210$.

Finding One Coefficient

- The coefficient we just calculated is the number of ways to select 16 objects with repetition from 5 types, with at least two from each type.
- We could have solved that problem by assigning two objects to each type and then counting the ways to select the other six. But the GF computation applied this "trick" for us, using only our normal intuition about polynomials.

Some Easy Counting Problems

- Let's say that we now want to collect \$15 from 20 distinct people. The first 19 people can give \$1 or nothing, and the last person can give \$1, \$5, or nothing. How many ways?
- It should be clear by now that we want the coefficient of x^{15} in $(1+x)^{19}(1+x+x^5)$.
- Let the x^r coefficient in $(1+x)^{19}$ be a_r, and let the x^r coefficient in $(1+x+x^5)$ be b_r. We want the x^{15} coefficient in the product.

Some Easy Counting Problems

- The x^{15} coefficient is the sum over all r of $a_r b_{15-r}$, which by the nature of the b's is just $a_{10} + a_{14} + a_{15}$, since most of the b's are 0.
- This is just C(19, 10) + C(19, 14) + C(19, 15).
- Again a breakdown into three cases based on the last person's gift would have gotten us this solution pretty quickly. But the generating function made this case analysis somewhat more automatic.

Some Easy Counting Problems

- How many ways are there to distribute 25 identical balls into seven distinct boxes, if box 1 has no more than 10 balls but the other boxes may have any number?
- The GF for this problem is $(1+x+\ldots+x^{10})(1+x+x^2+\ldots)^6 = (1-x^{11})/(1-x)$ times $1/(1-x)^6$, which we may write as $(1-x^{11})/(1-x)^7$.
- The x^{25} coefficient of the product of $1-x^{11}$ and $1/(1-x)^7$ is a sum of terms for i from 0 to 25, but most of those terms are 0.

Some Easy Counting Problems

- If we write $1/(1-x)^7$ as the sum of $b_r x^r$, the x^{25} coefficient of the product is $b_{25} - b_{14}$, which is $C(31, 25) - C(20, 14)$.
- Again there is a combinatorial interpretation. $C(31, 25)$ is the number of ways to put 25 identical balls into seven distinct boxes, and $C(20, 14)$ is the number of these distributions that have 11 or more balls in the first box.

A Harder Counting Problem

- Here is a problem that would be much more complicated without GF's, if we could manage it at all.
- How many ways are there to select 25 toys from seven types of toys, with between two and six of each type?
- The GF we want is $(x^2+\ldots+x^6)^7$, and we again want the x^{25} coefficient. This polynomial can be rewritten as $x^{14}(1+\ldots+x^4)^7$.

A Harder Counting Problem

- Of course the x^{25} coefficient in $x^{14}(1+\ldots+x^4)^7$ is the x^{11} coefficient in $(1-x^5)^7/(1-x)^7$.
- Writing this as $f(x)g(x)$, we then have that $f(x) = 1 - C(7, 1)x^5 + C(7, 2)x^{10} - C(7, 3)x^{15} + \ldots - C(7, 7)x^{35}$. And $g(x)$ is $1 + C(7, 1)x + C(8, 2)x^2 + C(9, 3)x^3 + \ldots + C(r+6, r)x^r + \ldots$
- Fortunately, only the first three terms of $f(x)$ can contribute to the x^{11} coefficient. By the product rule we have $C(7, 0)C(17, 11) - C(7, 1)C(12, 6) + C(7, 2)\,C(7, 1)$.

A Harder Counting Problem

- How would we have gotten C(7, 0)C(17, 11) - C(7, 1)C(12, 6) + C(7, 2)C(7, 1) by purely combinatorial reasoning?
- We start by taking two toys of each type and seeing how to distribute the other 11. C(17, 11) is the number of ways to do this with the restriction. C(7, 1)C(12, 6) is the number of ways to put five more of the 11 in one of the types, then distribute the other 6 arbitrarily.
- But this double counts a few cases where two types get 7 or more toys.

Verifying an Identity

- Tucker's final example in this section is to verify one of our binomial coefficient identities.
- We saw that $C(2n, n) = C(n, 0)^2 + C(n, 1)^2 + \ldots C(n, n)^2$, as we proved by block-walking.
- Let's look at the equation $(1+x)^{2n} = (1+x)^n \times (1+x)^n$. The coefficient of x^n in the LHS is clearly $C(2n, n)$. We can use our product rule to get the coefficient of x^n in the RHS.

Verifying an Identity

- The coefficient of x^n in a product is the sum, for all i, of the coefficient of x^i in the first factor times the coefficient of x^{n-i} in the second.
- This gives us $C(n, 0)C(n, n) + C(n, 1)C(n, n-1) + C(n, 2)C(n, n-2) + \ldots + C(n, n)C(n, 0)$.
- Since $C(n, k) = C(n, n-k)$ for any k, we can rewrite this as $C(n, 0)^2 + \ldots + C(n, n)^2$, giving us the RHS of our desired identity for $C(2n, n)$.
- Using the GF pretty much duplicates the reasoning from the block-walking proof.