CMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #19: Generating Function Models
(Tucker Section 6.1)
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Generating Function Models

- Idea of a Generating Function
- Power Series Examples
- The GF for $C(n, r)$
- Exploring $(1 + x + x^2)^4$
- Choosing Submultisets
- Limited and Unlimited Repetition
- Constrained Choices

The Idea of a GF

- A **generating function** is a single mathematical object that represents all the solutions to a parametrized counting problem.
- Let a_r be the number of ways to select r objects in a certain procedure. The generating function for a_r is the formal polynomial $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where n is the maximum number of objects we can select.

The Idea of a GF

- It may be that there is no upper limit to the number of objects we can select. In this case the terms go on forever, with a term $a_r x^r$ for every natural number r, and the generating function is a **formal power series**.
- Note that we don't consider replacing x with any value. What we normally do with a GF is to determine one of its coefficients, to answer the counting problem for some r.

Power Series Examples

- Any sequence a_r that has only finitely many nonzero entries is just a polynomial.
- Perhaps the simplest power series is the GF for the sequence that has $a_r = 1$ for all r. It is $1+x+x^2+x^3+\ldots$, which is also given by the **rational function** $1/(1-x)$. (A rational function is the *ratio* of two polynomials.)
- We can also have $1/(1-x^2) = 1+x^2+x^4+x^6+\ldots$, or $1/(1-kx) = 1 + kx + k^2x^2 + k^3x^3 + \ldots$, the GF for the sequence $a_r = k^r$.

Power Series Examples

- We can multiply two power series together to get another. For example, let's look at the rational function $1/(1-x)^2$, which is the series $1/(1-x) = 1+x+x^2+x^3+\ldots$ multiplied by itself.
- We have a term x^{i+j} for every pair of terms with x^i in the first copy and x^j in the second. There is one such term with $i+j = 0$, two with $i+j = 1$, three with $i+j = 2$, and so forth. The series is $1+2x+3x^2+4x^3+\ldots$, the GF for the sequence with $a_r = r+1$.

Multiplying Power Series

- If I have two GF's, one for the sequence a_r and one for the sequence b_r, I can make a third GF for a sequence c_r by multiplying the first two.

- This gives us $c_0 = a_0b_0$, $c_1 = a_0b_1 + a_1b_0$, $c_2 = a_0b_2 + a_1b_1 + a_2b_0$, and so forth with the general term c_r being the sum for i from 0 to r of a_ib_{r-i}.

- This is the number of ways to choose r objects, some in a's way and the rest in b's.

The GF for C(n, r)

- In our last lecture we noted that by the Binomial Theorem, $(1+x)^n$ is equal to the polynomial $C(n,0) + C(n,1)x + C(n, 2)x^2 + \dots + C(n,n)x^n$.
- This polynomial is a GF for the problem of choosing a binary string of length n with r 1's, or of choosing a subset of an n-element set with r elements. There are no terms after the one for x^n because there are no solutions to the counting problem after that point.

The GF for C(n, r)

- If we expand the product $(1+x)^n$ into 2^n different terms, the ones that simplify to x^r are exactly those with r copies of x and n-r copies of 1. Of course there are C(n, r) = C(n, n-r) of these.
- Each of these terms is a product of n factors, where each factor is x^0 or x^1. The term is thus x to the power $e_1+e_2+\ldots+e_n$, where each e_i is either 0 or 1.
- The number of such sums of e's totaling to r is the coefficient of x^r in our eventual polynomial.

Exploring $(1+x+x^2)^4$

- We can raise $(1+x+x^2)$ to the fourth power by first squaring it to get $(1+2x+3x^2+2x^3+x^4)$, then squaring that to get $(1+4x+10x^2+16x^3+19x^4+16x^5+10x^6+4x^7+x^8)$.
- The coefficient of x^r is the number of sums $e_1+e_2+e_3+e_4$ that total to r, where each e_i is equal to 0, 1, or 2. Note that the coefficients sum to $81 = 3^4$, the total number of terms.
- This is the GF for the number of ways to select r objects of four types, with ≤ 2 of each type.

Choosing Submultisets

- Consider the problem of choosing r balls from a collection of three red, three blue, three green and three yellow balls.
- This is choosing a **submultiset** of the given multiset of size 12.
- The GF for this problem is $(1+x+x^2+x^3)^4$. The x^r coefficient of this GF is the number of sums $e_1+e_2+e_3+e_4$ totaling to r, with each e_i at most 3.

Choosing Submultisets

- Similarly, we can easily form the GF for the number of multisets of each size that are contained within a given fixed multiset.
- Suppose we are to pick r donuts from a set of five chocolate, five strawberry, three lemon, and three cherry donuts. We must thus pick a sum $e_1+e_2+e_3+e_4 = r$ where the first two numbers are in the range from 0 through 5 and the last two are 0 through 3.

Choosing Submultisets

- We multiply two copies of $(1+x+x^2+x^3+x^4+x^5)$ and two copies of $(1+x+x^2+x^3)$, and get a polynomial that is the sum of a term for each 4-tuple of e's from the correct ranges. The coefficient of x^r is the number of such terms that have $e_1+e_2+e_3+e_4 = r$.
- What if we insist that we select at least one donut of each flavor? We can get a GF for this new problem by removing the 1 from each of the four factors, since 0 is no longer a valid exponent.

Limited and Unlimited Reps

- Remember that we can rephrase some of our counting problems in terms of distributing r objects from a set of n types, with either no repetition, limited repetition, or unlimited repetition.
- The GF for no repetition is just $(1+x)^n$, as we saw. If we can have up to t objects of each type, for example, we have $(1+x+\ldots+x^t)^n$.
- What about unlimited repetition?

Unlimited Repetition

- This case asks us how many sums there are of the form $e_1+\ldots+e_n = r$, with no restrictions on the e's. The GF for this is given by just $(1+x+x^2+x^3+\ldots)^n$, which we saw earlier is the same as $1/(1-x)^n$.
- If we are only looking for r total objects, we don't need to consider repeating a type more than r times. But it turns out to be simpler to use the power series with infinitely many terms, and deal with r only when we evaluate a coefficient.

Constrained Choices

- These techniques work to produce a GF whenever we have a distribution of identical objects into n boxes, with *any* restrictions on the number going into each box, as long as the box restrictions are independent of one another.
- For example, suppose we have two boxes, and we may put any even number of objects in the first and between three and five in the second.

Constrained Choices

- A GF for the first choice is $1/(1-x^2) = (1+x^2+x^4+\ldots)$, and for the second choice is $(x^3+x^4+x^5)$.
- We simply multiply this power series by this polynomial to get a power series that is a GF for the entire problem, given by the rational function $(x^3+x^4+x^5)/(1-x^2)$.
- Our next problem is to compute any given coefficient of a GF given in this way.