Lecture #19: Generating Function Models
(Tucker Section 6.1)
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Generating Function Models

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The Idea of a GF

- A generating function is a single mathematical object that represents all the solutions to a parametrized counting problem.

- Let $a_r$ be the number of ways to select $r$ objects in a certain procedure. The generating function for $a_r$ is the formal polynomial $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, where $n$ is the maximum number of objects we can select.
The Idea of a GF

• It may be that there is no upper limit to the number of objects we can select. In this case the terms go on forever, with a term $a_r x^r$ for every natural number $r$, and the generating function is a formal power series.

• Note that we don’t consider replacing $x$ with any value. What we normally do with a GF is to determine one of its coefficients, to answer the counting problem for some $r$. 
Power Series Examples

- Any sequence $a_r$ that has only finitely many nonzero entries is just a polynomial.

- Perhaps the simplest power series is the GF for the sequence that has $a_r = 1$ for all $r$. It is $1+x+x^2+x^3+\ldots$, which is also given by the rational function $1/(1-x)$. (A rational function is the ratio of two polynomials.)

- We can also have $1/(1-x^2) = 1+x^2+x^4+x^6+\ldots$, or $1/(1-kx) = 1 + kx + k^2x^2 + k^3x^3 + \ldots$, the GF for the sequence $a_r = k^r$. 
Power Series Examples

- We can multiply two power series together to get another. For example, let’s look at the rational function $1/(1-x)^2$, which is the series $1/(1-x) = 1+x+x^2+x^3+\ldots$ multiplied by itself.

- We have a term $x^{i+j}$ for every pair of terms with $x^i$ in the first copy and $x^j$ in the second. There is one such term with $i+j = 0$, two with $i+j = 1$, three with $i+j = 2$, and so forth. The series is $1+2x+3x^2+4x^3+\ldots$, the GF for the sequence with $a_r = r+1$. 
Multiplying Power Series

• If I have two GF’s, one for the sequence $a_r$ and one for the sequence $b_r$, I can make a third GF for a sequence $c_r$ by multiplying the first two.

• This gives us $c_0 = a_0b_0$, $c_1 = a_0b_1 + a_1b_0$, $c_2 = a_0b_2 + a_1b_1 + a_2b_0$, and so forth with the general term $c_r$ being the sum for $i$ from 0 to $r$ of $a_ib_{r-i}$.

• This is the number of ways to choose $r$ objects, some in a’s way and the rest in b’s.
The GF for $C(n,r)$

- In our last lecture we noted that by the Binomial Theorem, $(1+x)^n$ is equal to the polynomial $C(n,0) + C(n,1)x + C(n, 2)x^2 + \ldots + C(n,n)x^n$. 

- This polynomial is a GF for the problem of choosing a binary string of length $n$ with $r$ 1’s, or of choosing a subset of an $n$-element set with $r$ elements. There are no terms after the one for $x^n$ because there are no solutions to the counting problem after that point.
The GF for $C(n, r)$

• If we expand the product $(1+x)^n$ into $2^n$ different terms, the ones that simplify to $x^r$ are exactly those with $r$ copies of $x$ and $n-r$ copies of $1$. Of course there are $C(n, r) = C(n, n-r)$ of these.

• Each of these terms is a product of $n$ factors, where each factor is $x^0$ or $x^1$. The term is thus $x$ to the power $e_1+e_2+\ldots+e_n$, where each $e_i$ is either 0 or 1.

• The number of such sums of $e$'s totaling to $r$ is the coefficient of $x^r$ in our eventual polynomial.
Exploring $(1+x+x^2)^4$

- We can raise $(1+x+x^2)$ to the fourth power by first squaring it to get $(1+2x+3x^2+2x^3+x^4)$, then squaring that to get $(1+4x+10x^2+16x^3+19x^4+16x^5+10x^6+4x^7+x^8)$.

- The coefficient of $x^r$ is the number of sums $e_1+e_2+e_3+e_4$ that total to $r$, where each $e_i$ is equal to 0, 1, or 2. Note that the coefficients sum to $81 = 3^4$, the total number of terms.

- This is the GF for the number of ways to select $r$ objects of four types, with $\leq 2$ of each type.
Choosing Submultisets

• Consider the problem of choosing \( r \) balls from a collection of three red, three blue, three green and three yellow balls.

• This is choosing a **submultiset** of the given multiset of size 12.

• The GF for this problem is \((1+x+x^2+x^3)^4\). The \( x^r \) coefficient of this GF is the number of sums \( e_1+e_2+e_3+e_4 \) totaling to \( r \), with each \( e_i \) at most 3.
Choosing Submultisets

• Similarly, we can easily form the GF for the number of multisets of each size that are contained within a given fixed multiset.

• Suppose we are to pick $r$ donuts from a set of five chocolate, five strawberry, three lemon, and three cherry donuts. We must thus pick a sum $e_1 + e_2 + e_3 + e_4 = r$ where the first two numbers are in the range from 0 through 5 and the last two are 0 through 3.
Choosing Submultisets

• We multiply two copies of \((1+x+x^2+x^3+x^4+x^5)\) and two copies of \((1+x+x^2+x^3)\), and get a polynomial that is the sum of a term for each 4-tuple of e’s from the correct ranges. The coefficient of \(x^r\) is the number of such terms that have \(e_1+e_2+e_3+e_4 = r\).

• What if we insist that we select at least one donut of each flavor? We can get a GF for this new problem by removing the 1 from each of the four factors, since 0 is no longer a valid exponent.
Limited and Unlimited Reps

• Remember that we can rephrase some of our counting problems in terms of distributing \( r \) objects from a set of \( n \) types, with either no repetition, limited repetition, or unlimited repetition.

• The GF for no repetition is just \((1+x)^n\), as we saw. If we can have up to \( t \) objects of each type, for example, we have \((1+x+\ldots+x^t)^n\).

• What about unlimited repetition?
Unlimited Repetition

• This case asks us how many sums there are of the form $e_1 + \ldots + e_n = r$, with no restrictions on the $e$’s. The GF for this is given by just $(1 + x + x^2 + x^3 + \ldots)^n$, which we saw earlier is the same as $1/(1-x)^n$.

• If we are only looking for $r$ total objects, we don’t need to consider repeating a type more than $r$ times. But it turns out to be simpler to use the power series with infinitely many terms, and deal with $r$ only when we evaluate a coefficient.
Constrained Choices

• These techniques work to produce a GF whenever we have a distribution of identical objects into n boxes, with *any* restrictions on the number going into each box, as long as the box restrictions are independent of one another.

• For example, suppose we have two boxes, and we may put any even number of objects in the first and between three and five in the second.
Constrained Choices

• A GF for the first choice is $1/(1-x^2) = (1+x^2+x^4+\ldots)$, and for the second choice is $(x^3+x^4+x^5)$.

• We simply multiply this power series by this polynomial to get a power series that is a GF for the entire problem, given by the rational function $(x^3+x^4+x^5)/(1-x^2)$.

• Our next problem is to compute any given coefficient of a GF given in this way.