NAME: ____________________________

SPIRE ID: ____________________________

COMPSCI 250
Introduction to Computation
Final Exam Spring 2025

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DIRECTIONS:

- Answer the problems on the exam pages.
- There are **five** problems on pages **2-15**, some with multiple parts, for 125 total points plus 10 extra credit. Final scale will be determined after the exam.
- Page **16** contains useful definitions and is given to you separately – do not put answers on it!
- If you need extra space use the back of a page – both sides are scanned.
- But, if you do write a solution on the back, you must **explicitly** add a note on the front stating that you used the back page. Otherwise, we might not see your solution on Gradescope.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer.
- Your answers must be LEGIBLE, and not cramped. Write only short paragraphs with space between paragraphs.

1	**/35**
2	**/10+5**
3	**/20**
4	**/40+5**
5	**/20**
Total	**/125+10**

Question 1 (Dog Proof, 35 total points) With the warmer weather, Blaze and Rhonda have spent more time in their backyard, observing various animals and attempting to catch some of them.

Let $D = \{b, r\}$ be the set of dogs $\{\text{Blaze}, \text{Rhonda}\}$.

Let $A = \{\text{Bat}, \text{Cat}, \text{Crow}, \text{Rabbit}, \text{Robin}, \text{Skunk}, \text{Toad}, \text{Vole}\}$ be the set of animals they encountered.

Let F and M be two unary predicates on A such that $F(x)$ means "animal x can fly" and $M(x)$ means "animal x is a mammal". The set of mammals in A is $\{\text{Bat}, \text{Cat}, \text{Rabbit}, \text{Skunk}, \text{Vole}\}$ and the set of flying animals in A is $\{\text{Bat}, \text{Crow}, \text{Robin}\}$.

Finally, $C \subseteq D \times A$ is a binary predicate such that $C(d, a)$ means "dog d caught animal a".

(A, 10) Translations: Translate each statement as indicated.

- **Statement I:** (to symbols) Blaze did not catch a Skunk, and Blaze caught a Rabbit if and only if Blaze caught a Cat.

- **Statement II:** (to English) $C(b, \text{Cat}) \rightarrow \neg(C(b, \text{Skunk}) \vee C(b, \text{Rabbit}))$.

- **Statement III:** (to symbols) No dog caught any flying animal.

- **Statement IV:** (to English) $\forall x : [C(r, x) \rightarrow \neg M(x)] \wedge [C(b, x) \rightarrow M(x)]$

- **Statement V:** (to symbols) Each dog caught exactly one animal.

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(B, 10) Boolean Proof:

Using Statement I and II *only*, prove that the three boolean statements $p_1 = C(b, \text{Cat})$, $p_2 = C(b, \text{Rabbit})$, and $p_3 = C(b, \text{Skunk})$ are **all false**.

You may use either a truth table or a propositional proof. Remember that you must prove *both* that your solution satisfies Statements I and II, *and* that no other solution satisfies both of them.

Hint: When we say "Statements I and II only", this includes the fact that you should not use Statement V.

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(C, 15) Predicate Proof:

Using any or all of Statements I, II, III, IV, and V, **determine exactly which** animals, if any, were caught by each dog. Your argument should make clear, for each dog and each animal, whether that dog caught it.

Do this by filling out this table, with a "0" for "false" and a "1" for "true".

If your answer involves the use of a quantifier proof rule, and many of them should, make clear which rule you are using and when. Using complete sentences will generally make your answers more readable.

C(b, Bat)	=		C(r, Bat)	=	
C(b, Cat)	=	**0**	C(r, Cat)	=	
C(b, Crow)	=		C(r, Crow)	=	
C(b, Rabbit)	=	**0**	C(r, Rabbit)	=	
C(b, Robin)	=		C(r, Robin)	=	
C(b, Skunk)	=	**0**	C(r, Skunk)	=	
C(b, Toad)	=		C(r, Toad)	=	
C(b, Vole)	=		C(r, Vole)	=	

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Question 2 (A,10): (Induction 1)

For natural n, set $S_n = \sum_{i=1}^{n} \frac{1}{(2k-1)(2k+1)}$.

Prove by induction that, for all positive naturals n, $S_n = \frac{n}{2n+1}$.

As an example, $S_3 = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} = \frac{3}{7} = \frac{3}{2\cdot 3+1}$.

(i) First write your induction hypothesis in the box below. This should be in the form $P(x)$, where you *must* explicitly explain what x is and write an unambiguous statement of $P(x)$.

(ii) Next, write your base case(s) in the box below.

(iii) Finally, provide your induction step. This step will be marked on how clear and mathematically precise your proof is. Ambiguous explanations will have points deducted.
Be sure to clearly describe your induction goal and, in your induction step, exactly where the induction hypothesis is being used.

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Question 2 (B, 5):: (Induction Extra Credit)

Define A_n as follows:

$$A_0 = 0, \quad A_1 = 2, \quad \text{and } \forall n > 1, \quad A_{n+1} = 4A_n - 4A_{n-1}.$$

Note $A_2 = 4A_1 - 4A_0 = 4 \cdot 2 = 8$ and $A_3 = 4A_2 - 4A_1 = 4 \cdot 8 - 4 \cdot 2 = 24$.
So, $A_2 = 2 \cdot 2^2$ and $A_3 = 3 \cdot 2^3$.

Prove by induction that, for all naturals n, $A_n = n2^n$.

(i) First write your induction hypothesis in the box below. This should be in the form $P(x)$, where you *must* explicitly explain what x is and write an unambiguous statement of $P(x)$.

(ii) Next, write your base case(s) in the box below.

(iii) Finally, provide your induction step. This step will be marked on how clear and mathematically precise your proof is. Ambiguous explanations will have points deducted.
Be sure to clearly describe your induction goal and, in your induction step, exactly where the induction hypothesis is being used.

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Question 3 (20 points): (Induction 2) Define the language $\mathbf{B}$ on $\Sigma = \{a, b, c\}$ as follows.

String $\mathbf{w} \in \Sigma^*$ is in $\mathbf{B}$ if

R1: $\mathbf{w} = \lambda$ (the empty string) or

R2: $\mathbf{w} = abc$ or

R3: $\mathbf{w} = aa\mathbf{v}b$ where $\mathbf{v} \in \mathbf{B}$ or

R4: $\mathbf{w} = a\mathbf{v}bb$ where $\mathbf{v} \in \mathbf{B}$ or

R5: $\mathbf{w} = uv$ where $u \neq \lambda$, $v \neq \lambda$, $\mathbf{u} \in \mathbf{B}$ and $\mathbf{v} \in \mathbf{B}$.

R6: No other strings are in $\mathbf{B}$.

Examples: Let $\mathbf{w}_1 = aabcbb$, $\mathbf{w}_2 = aaabbb$, $\mathbf{w}_3 = abcabb$, $\mathbf{w}_4 = aaaaabb$.
Then $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathbf{B}$ and $\mathbf{w}_4 \notin \mathbf{B}$.

Define $N_a(\mathbf{w})$, $N_b(\mathbf{w})$, and $N_c(\mathbf{w})$, to be, respectively, the number of a's, b's and c's in $\mathbf{w}$. Also define $|\mathbf{w}| = N_a(\mathbf{w}) + N_b(\mathbf{w}) + N_c(\mathbf{w})$ to be the total number of characters in $\mathbf{w}$.

$N_a(\mathbf{w}_1) = 2$, $N_b(\mathbf{w}_1) = 3$, $N_c(\mathbf{w}_1) = 1$. $|\mathbf{w}_1| = 6$.
$N_a(\mathbf{w}_2) = 3$, $N_b(\mathbf{w}_2) = 3$, $N_c(\mathbf{w}_2) = 0$. $|\mathbf{w}_2| = 6$.
$N_a(\mathbf{w}_3) = 2$, $N_b(\mathbf{w}_3) = 3$, $N_c(\mathbf{3}_2) = 1$. $|\mathbf{w}_3| = 6$.
$N_a(\mathbf{w_4}) = 5$, $N_b(\mathbf{w}_4) = 2$, $N_c(\mathbf{w}_4) = 0$. $|\mathbf{w}_4| = 7$.

Parts (A), (B) and (C) are on the following pages.

Parts (B) and (C) require induction proofs. When writing their solutions you must use the the mathematical notation we provided above.

In the proofs, be sure to clearly describe your induction goal and, in your induction step, exactly where the induction hypothesis is being used.

(B) and (C) are being marked on how clear and mathematically precise the proof is. Ambiguous explanations or explanations missing details will have points deducted.

Also if your proofs have multiple pieces, place each piece in a separate paragraph with space between the paragraphs.

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(A, 4)

A *derivation* that $\mathbf{w} \in \mathbf{B}$ is a listing of the rules that show that $\mathbf{w} \in \mathbf{B}$, one item per line.

Each line in the derivation should justify the creation of a specific string in $\mathbf{B}$. A justification is a stament of the rules used and previous lines results referred to.

The final line in the derivation should be the justification of $\mathbf{w}$.

Here is a derivation that $\mathbf{w} = aabcabbbb \in \mathbf{B}$.
1. $\mathbf{v_1} = \lambda \in \mathbf{B}$. (R1)
2. $\mathbf{v_2} = a\,bb = a\,\mathbf{v_1}\,bb \in \mathbf{B}$. (R4)
3. $\mathbf{v_3} = abc \in \mathbf{B}$. (R2)
4. $\mathbf{v_4} = abc\,abb = \mathbf{v_2}\mathbf{v_1} \in \mathbf{B}$. (R5 and lines 2 and 3)
5. $\mathbf{w} = a\,abcabb\,bb = a\mathbf{v_4}bb \in \mathbf{B}$. (R4)

Give a derivation that shows that the string $\mathbf{w} = aaabcaabb$ is in $\mathbf{B}$.
Write your answers on the lines below (you shouldn't need all of the lines.)

1. ____________________
2. ____________________
3. ____________________
4. ____________________
5. ____________________
6. ____________________
7. ____________________
8. ____________________
9. ____________________

(B, 8) Prove by induction that, for every $\mathbf{w} \in B$, $|w|$ is divisible by 3, i.e., $(|w| \bmod 3) = 0$.

(i) First write your induction hypothesis in the box below. This should be in the form $P(x)$, where you *must* explicitly explain what x is and write an unambiguous statement of $P(x)$.

(ii) Next, write your base case(s) in the box below.

(iii) Finally, provide your induction step. This step will be marked on how clear and mathematically precise your proof is. Ambiguous explanations or explanations missing details will have points deducted.

(C, 8) Prove by induction that, for every $\mathbf{w} \in B$, $N_a(\mathbf{w}) \leq 2N_b(\mathbf{w})$.

(i) First write your induction hypothesis in the box below. This should be in the form $P(x)$, where you *must* explicitly explain what x is and write an unambiguous statement of $P(x)$.

(ii) Next, write your base case(s) in the box below.

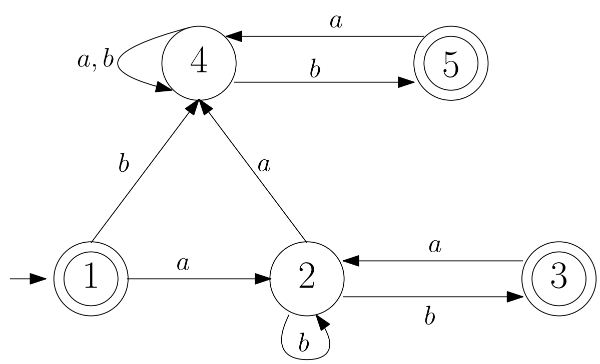
(iii) Finally, provide your induction step. This step will be marked on how clear and mathematically precise your proof is. Ambiguous explanations or explanations missing details will have points deducted.

Question 4 (40+5 points total):

This question involves several of the constructions from Kleene's Theorem.

We begin with an ordinary NFA N, with alphabet $\Sigma = \{a, b\}$, state set $\{1, 2, 3, 4, 5\}$, start state 1, final state set $\{1, 3, 5\}$, and transition relation

$$\{(1, a, 2), (1, b, 4), (2, b, 2), (2, b, 3), (2, a, 4), (3, a, 2), (4, a, 4), (4, b, 4), (4, b, 5), (5, a, 4)\}.$$



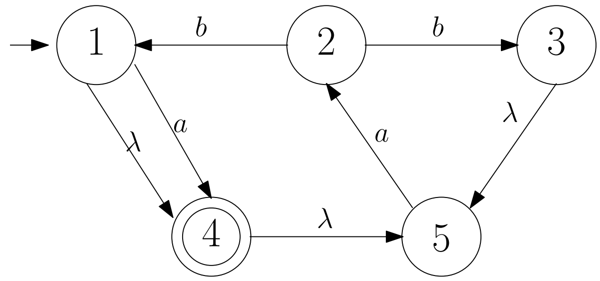
(a, 10) Subset Construction: *Using the Subset Construction*, find a DFA D that is equivalent to the NFA N. It's sufficient to show the new DFA diagram, without further explanation, if it comes from the given construction.

(b, 10) Minimization: Find a DFA D' that is minimal and has the same language as your DFA D from part (a). If you do not use the Minimization Construction, prove that your new DFA is minimal and that $L(D) = L(D')$.

(c, 10) State Elimination: Find and justify a regular expression that is equivalent to your DFA D and your minimized D'. If you use State Elimination on either DFA, no further correctness proof is required. If you use another method, prove that your regular expression is equivalent.

(d, 5XC) Making a λ-NFA: *Using the construction* from lecture and the textbook, compute a λ-NFA from the regular expression in part (c). A different equivalent λ-NFA will get only partial credit.

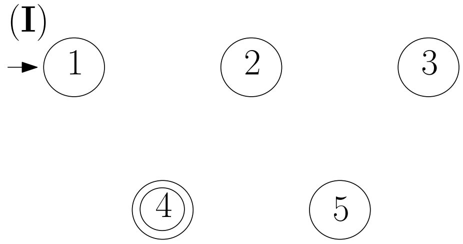
For the last part of this problem, we will work with a *different* language and machine. Here is a new λ-NFA K, which has alphabet $\Sigma = \{a, b\}$, state set $\{1, 2, 3, 4, 5\}$, start state 1, final state set $\{4\}$, and transition relation $(1, \lambda, 4), (1, a, 4), (2, b, 1), (2, b, 3), (3, \lambda, 5), (4, \lambda, 5), (5, a, 2)$.



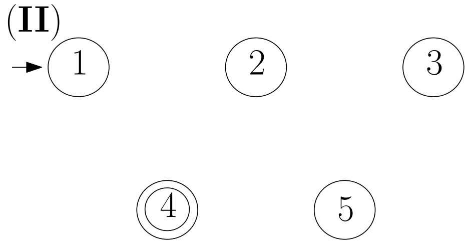
(e, 10) Killing λ-moves: *Using the construction* from the lectures and the textbook, find an ordinary NFA K' that is equivalent to the λ-NFA K above.

- In Graph (I), draw all the edges that are contained in the transitive closure of the λ-edges. Label them with λ.
- In Graph (II), draw all the a-letter moves for K'. Label them as a.
- In Graph (III), draw all the b-letter moves for K'. Label them as b.
- In Graph (IV) draw all the edges in K' properly labeled. Also properly denote the final states of K'.

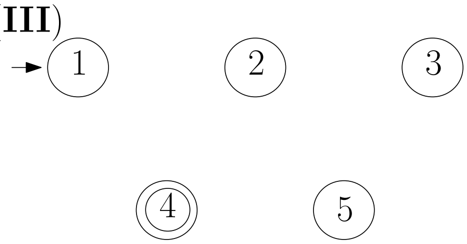
(I)



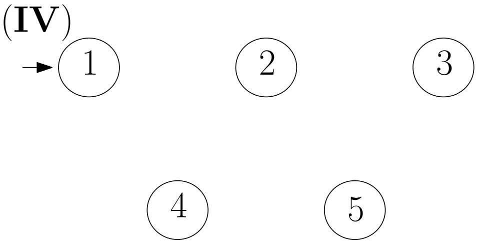
(II)



(III)



(IV)



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Question 5 (20): The following are ten true/false questions, with no explanation needed or wanted, no partial credit for wrong answers, and no penalty for guessing.

After reading the questions, write the correct answer, either T (for true) or F (for false), in the corresponding column. Be sure that your "T" and "F" characters are consistent and distinct.

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)

- (a) Statement V from the dog proof is equivalent to the statement that the relation C is a one-to-one function from D to A.
- (b) Let A and B be any two different non-empty finite sets. Then either $|A \cup B| > |A|$ or $|A \cup B| > |B|$, or both.
- (c) The regular expressions $(aba)^*ba$ and $ab(aba)^*$, over the alphabet $\{a, b\}$, denote the same language.
- (d) Let N be an NFA with at least one final state. Then it is possible that $L(N) = \emptyset$.
- (e) It is possible to have a DFA D and an NFA N such that $L(D) = L(N)$, but N has strictly fewer states than does D.
- (f) Let G be any connected undirected graph, with at least three nodes. Then there must exist two nodes a and b such that a DFS and BFS, with start node a and goal node b, will find different paths from a to b.
- (g) Let L_1 and L_2 be two languages over the same alphabet $\Sigma = \{a, b\}$. Then if $L_1 \subseteq L_2$ and L_2 is Turing decidable, then L_1 must also be Turing decidable.
- (h) Let $p > 1$ be a power of two and let $q > 1$ be an odd number. Then for any naturals a and b, if $a \equiv b \pmod{p}$ and $a \equiv b \pmod{q}$, then $a \equiv b \pmod{pq}$.
- (i) If the statements $p \rightarrow q$, $\neg q \rightarrow \neg r$, and $\neg(\neg p \wedge \neg r)$ all all true, then q must also be true.
- (j) Let M be a Turing machine whose transition function contains $\delta(i, \Box) = (p, a, R)$ and $\delta(p, a) = (p, a, L)$. (Here i is M's start state, $\Box$ is its blank symbol, and a is a letter in its input alphabet Σ.) Then it is possible that M halts on every input in Σ^*.

COMPSCI 250 Final Exam Supplementary Handout: 14 May 2025

With the warmer weather, Blaze and Rhonda have spent more time in their backyard, observing various animals and attempting to catch some of them.

Let $D = \{b, r\}$ be the set of dogs $\{\text{Blaze}, \text{Rhonda}\}$.

Let $A = \{\text{Bat}, \text{Cat}, \text{Crow}, \text{Rabbit}, \text{Robin}, \text{Skunk}, \text{Toad}, \text{Vole}\}$ be the set of animals they encountered.

Let F and M be two unary predicates on A such that $F(x)$ means "animal x can fly" and $M(x)$ means "animal x is a mammal". The set of mammals in A is $\{\text{Bat}, \text{Cat}, \text{Rabbit}, \text{Skunk}, \text{Vole}\}$ and the set of flying animals in A is $\{\text{Bat}, \text{Crow}, \text{Robin}\}$.

Finally, $C \subseteq D \times A$ is a binary predicate such that $C(d, a)$ means "dog d caught animal a".

- **Statement I:** (to symbols) Blaze did not catch a Skunk, and Blaze caught a Rabbit if and only if Blaze caught a Cat.
- **Statement II:** (to English) $C(b, \text{Cat}) \to \neg(C(b, \text{Skunk}) \vee C(b, \text{Rabbit}))$.
- **Statement III:** (to symbols) No dog caught any flying animal.
- **Statement IV:** (to English) $\forall x : [C(r, x) \to \neg M(x)] \wedge [C(b, x) \to M(x)]$
- **Statement V:** (to symbols) Each dog caught exactly one animal.

In question 3, string $\mathbf{w} \in \Sigma^*$ is in $\mathbf{B}$ if

R1: $\mathbf{w} = \lambda$ (the empty string) or

R2: $\mathbf{w} = abc$ or

R3: $\mathbf{w} = aa\mathbf{v}b$ where $\mathbf{v} \in \mathbf{B}$ or

R4: $\mathbf{w} = a\mathbf{v}bb$ where $\mathbf{v} \in \mathbf{B}$ or

R5: $\mathbf{w} = uv$ where $u \neq \lambda$, $v \neq \lambda$ $\mathbf{u} \in \mathbf{B}$ and $\mathbf{v} \in \mathbf{B}$.

R6: No other strings are in $\mathbf{B}$.

The NFA for Question 4(a) is repeated below: