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COMPSCI 250
Introduction to Computation
Second Midterm Spring 2023

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DIRECTIONS:

- Answer the problems on the exam pages.
- There are six problems on pages 2-10, some with multiple parts, for 100 total points plus 10 extra credit. Probable scale is somewhere around A=95, C=65, but will be determined after we grade the exam.
- If you need extra space use the back of a page.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer.

Question 1 (15): Recall that the Fibonacci function $F(n)$ is defined by the rules $F(0) = 0$, $F(1) = 1$, and for all n with $n > 1$, $F(n) = F(n-1) + F(n-2)$. Show that for all *positive* naturals n,

$$F(0)F(1) + F(1)F(2) + \ldots + F(2n-1)F(2n) = F(2n)^2.$$

Question 2 (25): Prove the two questions using induction.

- (a, 15) In the town of Gigilis, every year, exactly one child is born. Once the child is born, we say they are one year old. Every year the age of a Gigilis child multiplies by two (i.e. they become 8 when a typical child in a typical town is 3). You are visiting the town of Gigilis, and they ask how old you are. They don't understand the usual numbers, so you should express your age as the summation of the ages of the people in their town. For example, if Aigili is 1, Cigili is 4, and Eigili is 16 and you are 21 you should tell them that you are Eigili + Cigili + Aigili years old. Prove by induction that for any positive natural $n > 0$, any age n can be expressed in this way, assuming that there are enough children with the specified ages. Note that you cannot use someone's name twice.

- (b, 10, XC) The Gigilis children want to play a game. Each player chooses a binary string w, repeat it for i times (the concatenation of i copies of w when i is a non-negative integer), and the next player has to reverse it. For example, if the first player chooses $w = 001$ and $i = 4$, then will say "011011011011" and the next player has to say "110110110110". However, the children have a difficult time reversing a long string.

  Eigili, who recently turned 16 and started learning about induction, tells them they can reverse their string first and then repeat it i number of times instead. However, since Eigili learned the induction lesson recently, she cannot prove it and convince the others. Help Eigili by proving what she says is correct. In the other words prove by induction that for any string w and any natural i,

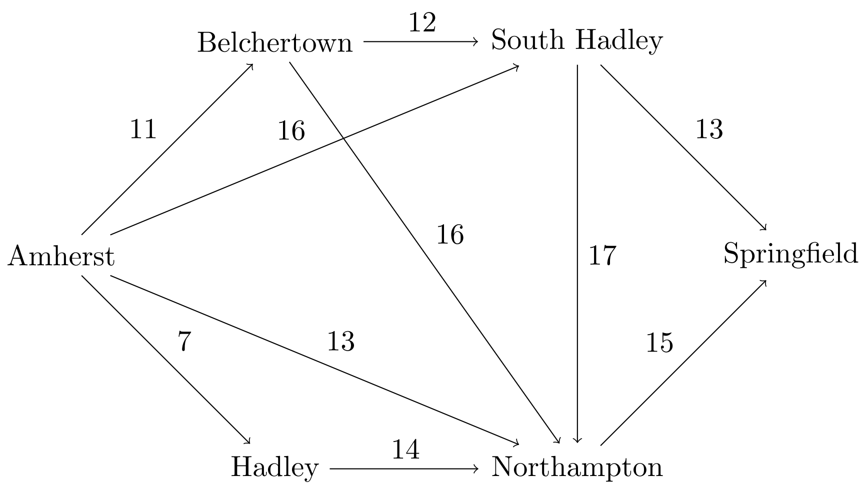
  $$(w^R)^i = (w^i)^R.$$

  (**Hint:** We have the definitions of reversal $\lambda^R = \lambda$ and $(wa)^R = aw^R$ and of exponentiation $w^0 = \lambda$ and $w^{i+1} = (w^i)w$. The rule $w(w^i) = w^{i+1}$ is not part of the definition, but we will let you use it without proof. Also, you may use the fact (proved in lecture) that for any strings u and v, $(uv)^R = v^R u^R$.)

Question 3 (10): Answer these two questions briefly.

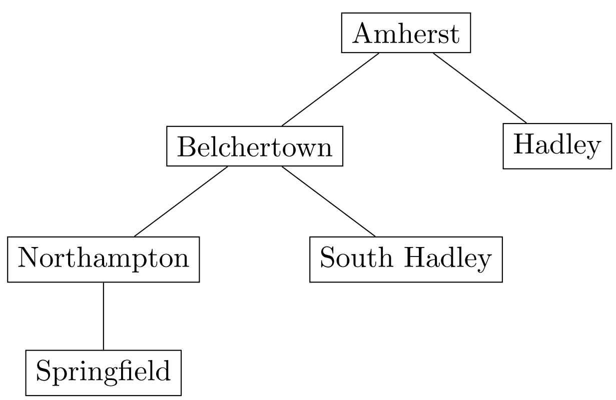
- (a, 5) Draw the parse tree for the formula $(p \wedge q) \vee (p \rightarrow (q \vee r))$ and find the postfix expression for this tree.
- (b, 5) We define the complement of an undirected graph G, as a graph $\overline{G}$ with the same vertex set and edge set as $\overline{E} = \{(x, y) | (x, y) \notin E\}$. How many edges exist in $\overline{G}$ when G has 8 nodes and 17 edges?

Question 4 (20): Pictured here is a directed graph G with six nodes. Its undirected version U is pictured on a following page. The weights are not used in this question.

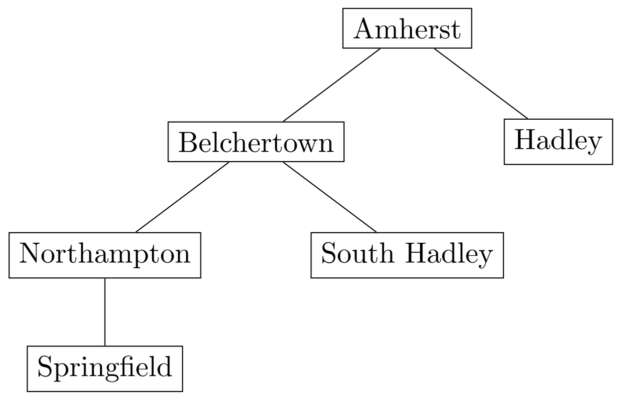


Answer the two following questions about the graphs G and U.

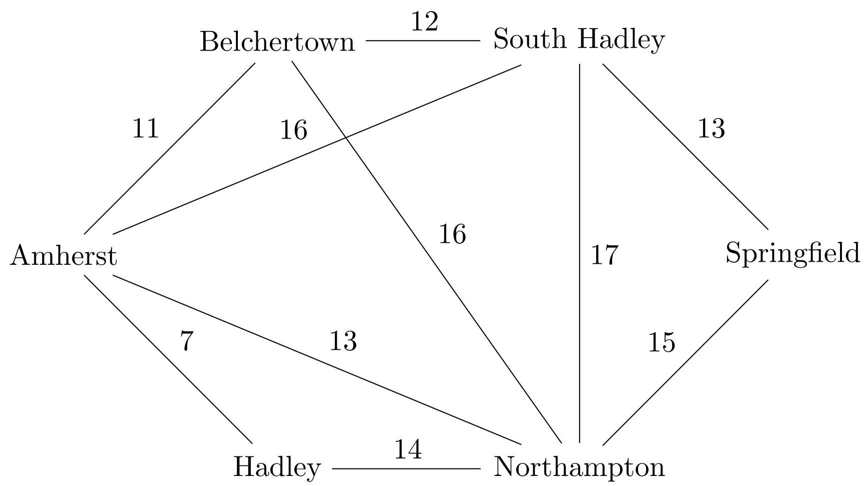
- (a, 10) Carry out a DFS search for the **directed** graph starting with node Amherst. When two or more nodes need to come off the stack, and they entered at the same time, take the one first that comes earlier alphabetically. Draw the DFS tree, indicating the non-tree edges, and classify each as a back, cross, or forward edge. Show each step of your stack, not only the final result.



- (b, 10) Ignore the edges' directions and conduct a BFS search for the *undirected* graph U, with Amherst as the start node. If two or more nodes need to come off the queue and they entered at the same time, take the one first that comes earlier alphabetically. Draw the BFS tree, indicating the non-tree edges. Show each step of your work, not only the final result.



Question 5 (20): Let U be the weighted undirected graph made from G, pictured here:



The six nodes in the graph represent locations in Massachusetts, and you are currently at Amherst, and you want to navigate from there to Springfield. The edges of the graph indicate traveling distance in miles to go from one location to another – for example, it takes 13 miles to drive directly from Amherst to Northampton.

Your goal is to reach Springfield with the minimum *total* distance.

The two questions follow on the next page.

- (a, 10) Trace a uniform-cost search with start node Amherst and no goal node, using the distances in miles given on the graph, to find the value of $d(x)$ for every node x. Indicate which nodes are on the priority queue at each stage of the search.

- (b, 10) In the undirected graph U we define a heuristic function h for the goal node Springfield. For every node x in the graph, $h(x)$ is 11 times the minimum number of edges in any path from x to Springfield. Trace an A^* search with start node Amherst and goal node Springfield, using h as the heuristic function. (You may assume without proof that h is an admissible and consistent heuristic for this search.) Indicate which nodes are on the priority queue at each stage of the search.

Question 6 (20): The following are ten true/false questions, with no explanation needed or wanted, no partial credit for wrong answers, and no penalty for guessing. Some of them refer to the scenarios of the other problems, and/or the entities defined on the supplemental sheet.

- (a) Let $\phi(x)$ be a predicate on the naturals. If $\phi(0)$ and $\phi(1)$ and $\phi(2)$ are true and we have $\forall x : \phi(x) \rightarrow (\phi(2x) \vee \phi(2x+1))$ being true, then $\forall x : \phi(x)$ is true.

- (b) The proof that strong induction is valid uses ordinary induction.

- (c) To compute the heuristic values in Question 5(b), we should use the breadth-first search from Question 4(b), multiplying the depth of each node by 11.

- (d) Let H be an undirected graph, and let us conduct a DFS from some node. If the resulting DFS tree contains at least one back edge, then there exists a cycle in H.

- (e) If T is any rooted tree with n nodes and e edges, then it is possible that $n = e + 2$.

- (f) The path relation in an undirected graph is always an equivalence relation.

- (g) The path relation in a directed graph is always a partial order relation.

- (h) The number of subsets of size 2 in a set S with n elements is $\frac{n^2}{2}$.

- (i) Let $P(x)$ be a property of **real** numbers. If we prove that $P(0)$ is true, and we prove that $\forall x : P(x) \rightarrow P(x+1)$, we may conclude that $P(x)$ is true for all real numbers x.

- (j) The relation $D(x, y)$, in a rooted tree, is defined so that $D(x, y)$ is true if and only if x is y's descendant. Then we can show that $D(x, y)$ is an equivalence relation.