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COMPSCI 250
Introduction to Computation
First Midterm Spring 2023

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DIRECTIONS:

- Answer the problems on the exam pages.
- There are four problems on pages 2-8, some with multiple parts, for 100 total points plus 5 extra credit. Final scale will be determined after the exam.
- Page 9 contains useful definitions and is given to you separately – do not put answers on it!
- If you need extra space use the back of a page.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer.

1	/15
2	/30
3	/15
4	/20+5
5	/20
Total	/100+5

Questions 1, 2, and 3 involve a set of four dogs $D = \{b, p, r, s\}$, named Blaze, Pushkin, Rhonda, and Scout, who one day all met at the park. Some of these dogs growled at some of the others.

We define a binary relation G on D, so that $G(x, y)$ means "dog x growled at dog y". (Pronoun note: All four of these dogs are female.)

Question 1 (15): Translate each statement as according to the directions:

- (a, 2) (to English) (Statement I) $G(p, s) \to G(b, p)$

- (b, 2) (to symbols) (Statement II) If Pushkin did not growl at Scout, then Pushkin growled at Rhonda but Blaze did not growl at Rhonda.

- (c, 2) (to English) (Statement III) $G(b, p) \to \neg G(b, p)$

- (d, 3) (to symbols) (Statement IV) It is not the case that some dog growled at herself.

- (e, 3) (to English) (Statement V) $\forall y : G(y, b) \to G(b, y)$

- (f, 3) (to symbols) (Statement VI) There is some dog that growled at every dog except herself.

Question 2 (30): These questions use the definitions, predicates, and premises on the supplementary sheet.

- (a, 10) Assuming *only* that Statements I, II and III are true, determine the truth values of the four propositions $q_1 = G(b, p)$, $q_2 = G(b, r)$, $q_3 = G(p, r)$, and $q_4 = G(p, s)$. You may use a truth table or a deductive sequence proof. Make sure that there is exactly one solution.

- (b, 20) Now assume that all of Statements I-VI are true. Using propositional and quantifier rules, prove that Blaze growled at some dog, that is, $\exists x : G(b, x)$ is true. You may use either English or symbols, but make it clear each time you use a quantifier proof rule.

Question 3 (15): Now using the same set of dogs D, and the relation G from Questions 1 and 2, we define a function $f : D \to \mathbf{N}$, where $\mathbf{N}$ is the set of all naturals $\{0, 1, 2, 3, \ldots\}$. For any dog x in D, we define $f(x)$ to be the number of dogs that x growled at, that is, $f(x) = |\{y : G(x, y)\}|$.

Let Statement VII be: "$\forall n : \exists x : f(x) = n$".

Let Statement VIII be "$\forall x : \forall y : (f(x) = f(y)) \to (x = y)$."

Here are your questions:

- (a, 2) What property of f is defined by Statement VII?

- (b, 3) Is Statement VII true of our function f defined above? Justify your answer.

- (c, 2) What property of f is defined by Statement VIII?

- (d, 8) Is Statement VIII true of our function f defined above? Justify your answer.

Question 4 (20+5): Here are a variety of number theory questions.

- (a, 5) Let $S = \{30, 31, 32, 33, 34, 35\}$ be a set of naturals. Which of these naturals have inverses modulo 42 and which do not? Justify your answers.

- (b, 10) Find inverses modulo 42 for every element of S that has one.

- (c, 5) **Cicadas** are large insects with unusual life cycles. When a cicada is born, it burrows underground for either 13 or 17 years (depending on the type of cicada), emerges after that time as an adult, and has its children in that year. A given adult cicada is normally part of a **brood**, all of whom emerged in the same year. Suppose that a given area there are both a single 13-year brood and a single 17-year brood of cicadas. What does the Chinese Remainder Theorem tell us about when the two broods will emerge in the same year?

- (d, 5XC) If our 13-year cicadas emerged in 2020, and our 17-year cicadas in 2023, and their cycles both persist through the entire coming millenium, what will be the first year after 3000 when both broods will emerge in the same year?

Question 5 (20): The following are ten true/false questions, with no explanation needed or wanted, no partial credit for wrong answers, and no penalty for guessing. Some of them use the sets, relations, and functions defined on the supplemental sheet, but you should assume the truth of Statements I-VII only if explicitly told to.

- (a) If X is the set $\{a, b, c\}$, then there are exactly three subsets of X that have exactly two elements.

- (b) $\emptyset^*$ is a non-empty language.

- (c) Statement IV in Questions 1-3 is equivalent to the statement "the relation G is not reflexive".

- (d) If a binary relation T, on the set A, is both an equivalence relation and a partial order, then A must be the empty set.

- (e) If two distinct positive naturals x and y are relatively prime, then at least one of them must be prime.

- (f) Let R be a binary relation on strings over the alphabet $\{a, b\}$, such that $R(u, v)$ is true if and only if the number of a's in u equals the number of b's in v. Then R is not reflexive, not symmetric, and not transitive.

- (g) Let S be a binary relation on the set $\{1, 2, 3, 4\}$ such that $S(x, y)$ is true if and only if $x \leq y$. Then S is a partial order, and its Hasse diagram has exactly four edges.

- (h) In a truth table with four boolean variables, for the compound proposition $C = p \wedge (q \vee r \vee s)$, there are exactly seven lines of the table in which C is true.

- (i) The statement $A \cup (B \cap C) = (A \cup B) \cap C$ is a set identity.

- (j) If n is an odd natural, and $n > 1$, then there exists an odd prime number p such that n is a multiple of p.

COMPSCI 250 First Midterm Supplementary Handout: 23 March 2023

Here are definitions of sets, predicates, and statements used on the exam.

Remember that the scope of any quantifier is always to the end of the statement it is in.

In Question 4 we often use the Java operator % on naturals, so that $x\%y$ is the remainder when x is divided by y.

The six statements of Question 1 are:

- (a, 2) (to English) (Statement I) $G(p, s) \rightarrow G(b, p)$
- (b, 2) (to symbols) (Statement II) If Pushkin did not growl at Scout, then Pushkin growled at Rhonda but Blaze did not growl at Rhonda.
- (c, 2) (to English) (Statement III) $G(b, p) \rightarrow \neg G(b, p)$
- (d, 3) (to symbols) (Statement IV) It is not the case that some dog growled at herself.
- (e, 3) (to English) (Statement V) $\forall y : G(y, b) \rightarrow G(b, y)$
- (f, 3) (to symbols) (Statement VI) There is some dog that growled at every dog except herself.