Lecture #9: Properties of Functions and Relations
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Relations and Functions

• Defining Functions With Quantifiers
• Total and Well-Defined Relations
• One-to-One and Onto Functions
• Bijections
• Composition and Inverse Functions
• Properties of Binary Relations on a Set
• Examples of Binary Relations on a Set
Relations and Direct Products

- Recall that when A and B are two sets, a **relation** from A to B is any set of ordered pairs, where the first element of each pair is from A and the second is from B.

- We say that the relation R is a subset of the **direct product** $A \times B$, which is the set of all such ordered pairs.
Functions

• A **function** in ordinary computing usage is an entity that gives an **output** of a given type (the **range**) whenever it is called with an input of a given type (the **domain**).

• A function **from A to B** takes input from A and gives output from B.

• A relation from A to B may or may not define a function from A to B.
We say that the relation is a function if for each input, there is exactly one possible output.

That is, for every element $x$ of $A$, there is exactly one element $y$ of $B$ such that the pair $(x, y)$ is in the relation.

We can put this definition into formal terms using predicates and quantifiers.
When a Relation is a Function

• Let R be a relation from A to B. We’ll write “(x, y) ∈ R” as “R(x, y)”, identifying the relation with its corresponding predicate. What does it mean for R to be a function?

• Part of the answer is that each x must have at least one y such that R(x, y) is true. In symbols, we say ∀x: ∃y: R(x, y). This property of a relation is called being total.
When a Relation is a Function

- We also require that each $x$ may have at most one $y$ such that $R(x, y)$ is true -- this is the property of being **well-defined**.

- We can write that no $x$ has more than one $y$, by saying $\forall x: \forall y: \forall z: (R(x, y) \land R(x, z)) \rightarrow (y = z)$.
  Another way to say this is $\neg \exists x: \exists y: \exists z: R(x, y) \land R(x, z) \land (y \neq z)$.

- A relation that is well-defined, but not necessarily total, is called a **partial function**. A non-void Java method computes a partial function since it may not terminate for all possible inputs.
Clicker Question #1

• Let $\mathbb{N}$ be the set of natural numbers \{0, 1, 2, 3, ...\}. Here are four binary relations on $\mathbb{N}$. Which one is a function from $\mathbb{N}$ to $\mathbb{N}$? Remember that a function must be both total and well-defined.

• (a) $A(x, y) = \{(x, y): x^2 + y^2 = 25\}$
• (b) $B(x, y) = \{(x, y): x^2 = y\}$
• (c) $C(x, y) = \{(x, y): x = 2y + 3\}$
• (d) $D(x, y) = \{(x, y): y = 6x - 4\}$
Answer #1

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• (d) \( D(x, y) = \{(x, y): y = 6x - 4\} \)
Onto Functions (Surjections)

- We can also use quantifiers to define two important properties of functions.
- A function is onto (also called a surjection) if every element of the range is the output for at least one input, in symbols $\forall y: \exists x: R(x, y)$. Note that this is not the same as the definition of “total” because the $x$ and $y$ are switched -- it is the dual property of being total.
Onto Functions (Surjections)

Onto, not 1-1

Not Onto

Onto and 1-1
One-to-One Functions

- A function is **one-to-one** (an **injection**) if it is never true that two different inputs map to the same output.

- We can write this as $\forall w: \forall x: \forall y: (R(w, y) \land R(x, y)) \rightarrow (w = x)$, or equivalently $\neg \exists w: \exists x: \exists y: R(w, y) \land R(x, y) \land (w \neq x)$. This is obtained from the well-defined property by switching domain and range.
One-to-one Functions (Injections)

1-1, not onto

Not 1-1

Onto and 1-1
These properties are important in **combinatorics** -- if \( A \) and \( B \) are finite sets, we can have a surjection from \( A \) to \( B \) if and only if \( |A| \geq |B| \).

Similarly, we can have an injection from \( A \) to \( B \) if and only if \( |A| \leq |B| \).

(Here "|A|" denotes the number of elements in \( A \), and "|B|" the number in \( B \).)
Bijections

- It is possible for a function to be both onto and one-to-one. We call such a function a **bijection** (also sometimes a **one-to-one correspondence** or a **matching**).

- From what we just said about the sizes of finite sets in a surjection or injection, we can see that a bijection from \(A\) to \(B\) is possible if and only if \(|A| = |B|\).
Clicker Question #2

• Suppose that A and B are two finite sets, and that f is a function from A to B. If f is onto but not one-to-one, what can we conclude about the sizes |A| and |B| of A and B?
  • (a) |A| ≤ |B|
  • (b) |A| = |B|
  • (c) |A| > |B|
  • (d) |A| ≥ |B|
Answer #2

• Suppose that A and B are two finite sets, and that f is a function from A to B. If f is onto but not one-to-one, what can we conclude about the sizes |A| and |B| of A and B?

• (a) |A| ≤ |B|
• (b) |A| = |B|
• (c) |A| > |B|
• (d) |A| ≥ |B|
• There is an interesting theory, which we don’t have time for in this course, about the sizes of infinite sets, where we define two sets to have the same “size” if there is a bijection from one to the other.

• A bijection from a set to itself is also called a permutation. The problem of sorting is to find a permutation of a set that puts it in some desired order.
Composition of Functions

- If $f$ is a function from $A$ to $B$, and $g$ is a function from $B$ to $C$, we can define a function $h$ from $A$ to $C$ by the rule $h(x) = g(f(x))$. We map $x$ by $f$ to some element $y$ of $B$, then map $y$ by $g$ to an element of $C$. This new function is called the composition of $f$ and $g$, and is written “$g \circ f$”.

- The notation $g \circ f$ is chosen so that $(g \circ f)(x) = g(f(x))$, that is, the order of $f$ and $g$ remains the same in these two ways of writing it.
Inverse Functions

- With quantifiers, we can define \((g \circ f)(x) = z\) to mean \(\exists y: (f(x) = y) \land (g(y) = z)\).

- If \(A\) and \(C\) are the same set, it is possible that the function \(g\) undoes the function \(f\), so that \(g(f(x))\) is always equal to \(x\). This can only happen when \(f\) is a bijection -- in this case \(A\) and \(B\) have the same size, and \(g\) must also be a bijection. We then say that \(f\) and \(g\) are inverse functions for each other.
Properties of Binary Relations

- Binary relations from a set to itself (called **relations on a set**) may or may not have certain properties that we also define with quantifiers.

- A relation $R$ is **reflexive** if $\forall x: R(x, x)$ is true, and **antireflexive** if $\forall x: \neg R(x, x)$. Note that “antireflexive” is not the same thing as “not reflexive”.
More Properties

- R is **symmetric** if $\forall x: \forall y: R(x, y) \rightarrow R(y, x)$, or equivalently $\forall x: \forall y: R(x, y) \leftrightarrow R(y, x)$.

- R is **antisymmetric** if $\forall x: \forall y: (R(x, y) \land R(y, x)) \rightarrow (x = y)$. Again “antisymmetric” is a different property from “not symmetric”.

- R is **transitive** if $\forall x: \forall y: \forall z: (R(x, y) \land R(y, z)) \rightarrow R(x, z)$. We saw this property in the last lecture with the “smaller than” property for dogs.
Examples of Binary Relations

- The **equality relation** $E$ is defined so that $E(x, y)$ is true if and only if $x = y$.
- This relation is reflexive, symmetric, and transitive.
- We’ll soon see that any relation with these three properties, called an **equivalence relation**, acts in many ways like equality.
Examples of Binary Relations

• On numbers, for example, we can define LE(x, y) to mean x ≤ y, and LT(x, y) to mean x < y.

• LE is reflexive, antisymmetric, and transitive, and relations with those three properties are called **partial orders**.

• LT, on the other hand, is antireflexive, antisymmetric, and transitive.
Examples of Binary Relations

- In the game of rock-paper-scissors, we can define a “beats” relation so that $B(x, y)$ means “$x$ beats $y$ in the game”.

- So $B(r, s)$, $B(s, p)$, and $B(p, r)$ are true and the other six possible atomic statements are false.

- This relation is antireflexive, antisymmetric, and not transitive.
Rock-Paper-Scissors

antireflexive -- no loops
antisymmetric -- no two-way arrows
not transitive -- two-step paths with no shortcuts
Clicker Question #3

- Let the binary relation $R$ on $\mathbb{N}$ be defined so that $R(x, y)$ is $\{(x, y) : x + 3 < y\}$. This relation is:
  - (a) both reflexive and transitive
  - (b) reflexive but not transitive
  - (c) transitive but not reflexive
  - (d) neither reflexive nor transitive
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