# CMPSCI 250 Discussion #6: The Truth Game
## Individual Handout

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We've just learned the four proof rules for quantified statement: Existential and Universal Instantiation (EI and UI) and Existential and Universal Generalization (EG and UG). Today we're going to practice some proofs with these rules, and look at the **game semantics** of quantified statements.

Given a quantified statement with no free variables, we can design a **Truth Game**, where two players called White and Black alternate moves, and White has a winning strategy if and only if the quantified statement is true. We want the statement to be in **prenex normal form**, meaning that the quantifiers all come at the beginning with no intervening statements or operators.

The rules of the game are simple. There is a move for each quantifier, in order from the beginning of the statement. If the quantifier is $\exists x :$, White names a value of the variable $x$, of the correct type. If the quantifier is $\forall x :$, Black names the value of $x$. When all the variables are set, the remainder of the statement is evaluated with those values. White wins if it is true and Black wins if it is false.

Here's the domain for our examples. The type of all variables is $S$, a set $\{a, b, c, d\}$ containing the four dogs Ace, Biscuit, Cardie, and Duncan. We can use the order operators $<$, $\leq$, $=$, $\geq$, and $>$, and these refer to alphabetical order on the dogs' names, so that $a < b < c < d$. All the usual rules for inequalities hold. There are two more predicates: $L(x)$ meaning "$x$ is large" and $M(x)$ meaning "$x$ is mine". Of these dogs, all but Duncan are large and only Cardie and Duncan are mine.

Let's play the game for $\exists x : M(x) \wedge L(x)$, "there exists a dog that is both large and mine". This statement is true, so White should be able to win. And she can, by using her only move to set $x$ to $c$. We then evaluate $M(c) \wedge L(c)$, which is true by the given values of the predicates. White had other moves that would lose the game, but if she makes the right move she wins.

This strategy corresponds to a formal proof of the statement using the EG rule. We derive $M(c) \wedge L(c)$ from the given values by Conjunction, then use EG to get $\exists x : M(x) \wedge L(x)$.

For a second example, look at $\forall x : \exists y : (x \neq y) \wedge M(y)$, "for every dog there is some different dog that is mine". Since this is true, White should have a winning strategy. The game begins with Black choosing some dog $x$. What dog should White pick for $y$? If $x$ is Cardie, she can pick Duncan, and if $x$ is Duncan, she can pick Cardie. If $x$ is not mine, either of my dogs will do. A simple strategy is "if $x = d$, let $y = c$, otherwise let $y = d$". With these choices of $y$, both $x \neq y$ and $M(y)$ will be true, and White will win.

How does this turn into a proof of the statement using the quantifier rules? We say "Let $x$ be an arbitrary dog. If $x = d$, let $y = c$, otherwise let $y = d$. The statement $(x \neq y) \wedge M(y)$ is then true. Since we found a $y$ making it true, we derive $\exists y : (x \neq y) \wedge M(y)$ by EG. Since $x$ was arbitrary, we have proved $\forall x : \exists y : (x \neq y) \wedge M(y)$ by UG."

Here are three statements for you to prove. In each case, play the Truth Game for the statement enough times that you are sure which player has a winning strategy, and what it is. If White has the winning strategy, write down a proof of the statement using the quantifier rules. If Black has the strategy, write down a proof of the *negation* of the statement. (You can compute the negation of the statement using the DeMorgan Laws for quantifiers.) (Note also that I am using my own scoping rules rather than Rosen's – any quantifier is in scope for the entire statement in which it appears unless it is enclosed in parentheses which these aren't.)

- Every dog in the set is either mine or comes strictly before Biscuit. ($\forall x : M(x) \vee (x < b)$).
- There is a dog $x$ such that if $y$ is any of my dogs, there is a third dog $z$ that comes strictly between $x$ and $y$. ($\exists x : \forall y : \exists z : M(y) \rightarrow ((x < z < y) \vee (y < z < x))$)
- Given any large dog, there is a second dog that is mine and is either large or comes strictly after the first dog, and there is a third dog that comes strictly before the first dog. ($\forall x : \exists y : \exists z : L(x) \rightarrow [M(y) \wedge (L(y) \vee (x < y)) \wedge (z < x)]$.)