CMPSCI 250: Introduction to Computation

Lecture #9: Properties of Functions and Relations
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Properties of Functions and Relations

- Defining Functions With Quantifiers
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- One-to-One and Onto Functions
- Bijections
- Composition and Inverse Functions
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Defining Functions With Quantifiers

- Recall that when A and B are two sets, a **relation** from A to B is any set of ordered pairs, where the first element of each pair is from A and the second is from B. We say that the relation R is a subset of the **direct product** $A \times B$.

- A **function** in ordinary computing usage is an entity that gives an **output** of a given type (the range) whenever it is called with an **input** of a given type (the domain). A function from A to B takes input from A and gives output from B.

- A relation from A to B may or may not define a function from A to B. We say that the relation is a function if for each input, there is *exactly one* possible output. That is, for every element x of A, there is exactly one element y of B such that the pair (x, y) is in the relation.

- We can put this definition into formal terms using predicates and quantifiers.

Total and Well-Defined Relations

- Let R be a relation from A to B. We'll write "(x, y) $\in$ R" as "R(x, y)", identifying the relation with its corresponding predicate. What does it mean for R to be a relation?

- Part of the answer is that each x must have *at least one* y such that R(x, y) is true. In symbols, we say $\forall x: \exists y: R(x, y)$. This property of a relation is called being **total**.

- The other part is that each x may have *at most one* y such that R(x, y) is true -- this is the property of being **well-defined**. We can write that no x has more than one y, by saying $\forall x: \forall y: \forall z: (R(x, y) \wedge R(x, z)) \rightarrow (y = z)$. Another way to say this is $\neg\exists x: \exists y: \exists z: R(x, y) \wedge R(x, z) \wedge (y \neq z)$.

- A relation that is well-defined, but not necessarily total, is called a **partial function**. A non-void Java method computes a partial function since it may not terminate for all possible inputs.

One-to-One and Onto Functions

- We can also use quantifiers to define two important properties of functions.

- A function is **onto** (also called a **surjection**) if every element of the range is the output for at least one input, in symbols $\forall y: \exists x: R(x, y)$. Note that this is not the same as the definition of total because the x and y are switched -- it is the **dual** property of being total.

- A function is **one-to-one** (an **injection**) if it is never true that two different inputs map map to the same output. We can write this as $\forall w: \forall x: \forall y: (R(w, y) \wedge R(x, y)) \rightarrow (w = x)$, or equivalently $\neg\exists w: \exists x: \exists y: R(w, y) \wedge R(x, y) \wedge (w \neq x)$. This is obtained from the well-defined property by switching domain and range.

- These properties are important in combinatorics -- if A and B are finite sets, we can have a surjection from A to B if and only if $|A| \geq |B|$, and we can have an injection from A to B if and only if $|A| \leq |B|$. (Here "$|A|$" denotes the number of elements in A, and "$|B|$" the number in B.)

Bijections

- It is possible for a function to be *both* onto and one-to-one. We call such a function a **bijection** (also sometimes a **one-to-one correspondence** or a **matching**).

- From what we just said about the sizes of finite sets in a surjection or injection, a bijection from A to be is possible if and only if |A| = |B|.

- There is an interesting theory, which we don't have time for in this course, about the sizes of *infinite* sets, where we *define* two sets to have the same "size" if there is a bijection from one to the other.

- A bijection from a set to itself is also called a **permutation**. The problem of **sorting** is to find a permutation of a set that puts it in some desired order.

Composition and Inverse Functions

- If f is a function from A to B, and g is a function from B to C, we can define a function h from A to C by the rule h(x) = g(f(x)). We map x by f to some element y of B, then map y by g to an element of C. This new function is called the **composition** of f and g, and is written "$g \circ f$".

- The notation $g \circ f$ is chosen so that $(g \circ f)(x) = g(f(x))$, that is, the order of f and g remains the same in these two ways of writing it.

- With quantifiers, we can define $(g \circ f)(x) = z$ to mean $\exists y: (f(x) = y) \wedge (g(y) = z)$.

- If A and C are the same set, it is possible that the function g *undoes* the function f, so that g(f(x)) is always equal to x. This can only happen when f is a bijection -- in this case A and B have the same size, and g must also be a bijection. We then say that f and g are **inverse functions** for each other.

Properties of Binary Relations on a Set

- Relations from a set to itself (called **relations on a set**) may or may not have certain properties that we also define with quantifiers.
- A relation R is **reflexive** if $\forall x$: R(x, x) is true, and **antireflexive** if $\forall x$: $\neg$R(x, x). Note that “antireflexive” is not the same thing as “not reflexive”.
- R is **symmetric** if $\forall x$: $\forall y$: R(x, y) $\rightarrow$ R(y, x), or equivalently $\forall x$: $\forall y$: R(x, y) $\leftrightarrow$ R(y, x). It is **antisymmetric** if $\forall x$: $\forall y$: (R(x, y) $\wedge$ R(y, x)) $\rightarrow$ (x = y). Again “antisymmetric” is a different property from “not symmetric”.
- R is **transitive** if $\forall x$: $\forall y$: $\forall z$: (R(x, y) $\wedge$ R(y, z)) $\rightarrow$ R(x, z). We saw this property last lecture with the “smaller than” property for dogs.

Examples of Binary Relations on a Set

- The **equality relation** E is defined so that E(x, y) is true if and only if x = y. This relation is reflexive, symmetric, and transitive. We'll soon see that any relation with these three properties, called an **equivalence relation**, acts in many ways like equality.

- On numbers, for example, we can define LE(x, y) to mean x ≤ y, and LT(x, y) to mean x < y. LE is reflexive, antisymmetric, and transitive, and relations with *those* three properties are called **partial orders**. LT is antireflexive, antisymmetric, and transitive.

- In the game of **rock-paper-scissors**, we can define a "beats" relation so that B(x, y) means "x beats y in the game". So B(r, s), B(s, p), and B(p, r) are true and the other six possible atomic statements are false. This relation is antireflexive, antisymmetric, and *not* transitive.