NAME: ______________________

SPIRE ID: ______________________

COMPSCI 250
Introduction to Computation
Final Exam Fall 2025

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DIRECTIONS:

- Answer the problems on the exam pages.
- There are **five** problems on pages **2-16**, some with multiple parts, for 120 total points plus 10 extra credit. Final scale will be determined after the exam.
- Page **17** contains useful definitions and is given to you separately – do not put answers on it!
- If you need extra space use the back of a page – both sides are scanned.
- But, if you do write a solution on the back, you must **explicitly** add a note on the front stating that you used the back page. Otherwise, we might not see your solution on Gradescope.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer.
- Your answers must be LEGIBLE, and not cramped. Write only short paragraphs with space between paragraphs.

1	**/35**
2	**/10**
3	**/15+5**
4	**/40+5**
5	**/20**
Total	**/120+10**

Family Name: ______________________

Question 1 (Dog Proof, 35 total points) At Thanksgiving dinner, there were nine members of Dave's extended family. Dave and his wife Jessica were joined by his daughter Julia, son-in-law Will, grandson Graham, granddaughter Genevieve, his own dogs Blaze and Rhonda, and his daughter's dog Gwen.

Let $F = \{B, D, Ge, Gr, Gw, Je, Ju, R, W\}$ be the family members, and let $D = \{B, Gw, R\}$ be a subset of F containing the dogs. The menu included Dog Food, Green Bean Casserole, Pie, and Turkey — we will consider the set $M = \{DF, GBC, P, T\}$ of those items. Let $A \subseteq F \times M$ be a binary predicate so that $A(x, y)$ means "family member x ate menu item y".

We have six statements about the relation A:

(a, 10) Translations: Translate each statement as indicated.

- **Statement I:** (2, to symbols) Exactly one family member ate all four menu items.

- **Statement II:** (1, to English) $A(Gr, GBC) \rightarrow (A(R, GBC) \wedge A(W, GBC))$.

- **Statement III:** (1, to symbols) If either Will did not eat GBC or Rhonda ate GBC, then Graham ate GBC and Rhonda did not eat GBC.

- **Statement IV:** (2, to English) $\forall x : \forall y : \forall z : [(x \in D) \wedge (y \in D)] \rightarrow (A(x, z) \leftrightarrow A(y, z))$.

- **Statement V:** (2, to symbols) Dave, Julia, and Will each ate exactly the same menu items.

- **Statement VI:** (2, to English) $\forall x : A(x, T) \leftrightarrow (x \neq Je)$.

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(b, 10) Boolean Proof:

Using Statement II and III *only*, prove that of the three boolean statements $p_1 = A(Gr, GBC)$, $p_2 = A(R, GBC)$, and $p_3 = A(W, GBC)$, p_1 and p_2 **are false** and that p_3 **is true**.

You may use either a truth table or a propositional proof. Please, for our ease in grading, use p_1, p_2, and p_3 rather than the original statements – you should write down your translation into this format before you start working.

Remember that you must prove *both* that your solution satisfies Statements II and III, *and* that no other solution satisfies both of them. Both follow from a complete truth table, but in a propositional proof you must make sure that both are included in your argument.

(c, 15) Predicate Proof:

Using any or all of Statements I, II, III, IV, V, and VI, **prove that Genevieve ate Dog Food**. We are providing a table for all 36 possible truth values of the relation A, but you are *not* asked to fill out this table completely, and we *do not* guarantee that all 36 values are determined from the six statements. We have already filled in the three facts you know from part (B), where **0** means false, and **1** means true.

$x \backslash y$	DF	GBC	P	T
B				
D				
Ge				
Gr		**0**		
Gw				
Je				
Ju				
R		**0**		
W		**1**		

If your answer involves the use of a quantifier proof rule, and many of them should, make clear which rule you are using and when. Using complete sentences will generally make your answers more readable. If your later statements follow from previous statements you have proved, make clear which is the premise and which the conclusion. Quantifier rules may be expressed in either symbols or English. If you are using your own translation of one of the premises in your proof, check again that your statement is equivalent to the original.

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Question 2 (10 points): (Induction 2) For naturals n, let A_n be defined by

$$A_0 = 2, \quad A_1 = 6 \quad \text{and for all } n \geq 1,\ A_{n+1} = 6A_n - 8A_{n-1}.$$

Examples: $A_2 = 6A_1 - 8A_0 = 36 - 16 = 20$ and $A_3 = 6A_2 - 8A_1 = 6 \cdot 20 - 8 \cdot 6 = 72$.

Prove *by induction*, that for *all* naturals n, $A_n = 4^n + 2^n$.

As a validity check. note that $A_2 = 20 = 4^2 + 2^2$ and $A_3 = 72 = 64 + 8 = 4^3 + 2^3$.

a) First write your induction hypothesis in the box below. This should be in the form $P(x)$, where you *must* explicitly explain what x is and write an unambiguous statement of $P(x)$.

(b) Next, write your base case(s) in the box below.

(c) Finally, provide your induction step. This step will be marked on how clear and mathematically precise your proof is. Ambiguous explanations or explanations missing details will have points deducted.

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Question 3 (a,5): (Induction 1)

We will say that a word w contains a *doubled letter* if some character in the word appears twice in a row. For example $w_1 = abccababa$ contains the doubled letter cc, while $w_2 = abcabcba$ does not contain any doubled letter.

For a fixed alphabet Σ let language NDL (no doubled-letter) be the set of all words in Σ^* that do not contain a double letter. For our problem, fix $\Sigma = \{a, b\}$, then

$$NDL = \{\lambda, a, b, ab, ba, aba, bab, abab, baba, ababa, babab, \ldots\}.$$

Furthermore, for all positive naturals n, define

$$A_n = \{w : w \in NDL,\ |w| = n, \text{ and } w \text{ ends in } a\} \quad \text{and} \quad B_n = \{w :\ w \in NDL,\ |w| = n,, \text{ and } w \text{ ends in } b\}.$$

As examples, $A_2 = \{ba\}$, $B_2 = \{ab\}$, $A_3 = \{aba\}$, $B_3 = \{bab\}$.
Recall that if X is a set, $|X|$ is the number of items in the set.

Let $P(n)$ be the statement "$|A_n| = |B_n| = 1$", defined for all positive naturals n.

Prove by induction that, for all positive naturals n, $P(n)$ is true.

Note that, for this problem, we have already given you the induction hypothesis, $P(n)$.

(i) Now, write your base case(s) in the box below.

(ii) Finally, provide your induction step, i.e. proving that assuming $P(n)$ implies $P(n + 1)$.

This step will be marked on how clear and mathematically precise your proof is. Ambiguous explanations will have points deducted.
Be sure to clearly describe your induction goal. If you run out of space, continue the proof on the back page (with a note stating that you are writing on the back).

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Question 3 (b, 10): (Induction)

Now let $\Sigma = \{a, b, c\}$. Then

$$NDL = \{\lambda, a, b, c, ba, ca, ab, cb, ac, bc, aba, cba, aca, bca, bab, cab, acb, bcb, bac, cac, abc, cbc, \ldots\}.$$

Furthermore, for all positive naturals n, define

$$A_n = \{w : \ w \in NDL,\ |w| = n, \text{ and } w \text{ ends in } a\}, \quad B_n = \{w : \ w \in NDL,\ |w| = n, \text{ and } w \text{ ends in } b\},$$

$$C_n = \{w : \ w \in NDL,\ |w| = n, \text{ and } w \text{ ends in } c\},$$

As examples, $A_2 = \{ca, ba\}$, $B_2 = \{ab, cb\}$, $C_2 = \{ac, bc\}$.

Prove by induction that, for all positive naturals n, $|A_n| = |B_n| = |C_n| = 2^{n-1}$.

(i) First write your induction hypothesis in the box below. This should be in the form $P(x)$, where you *must* explicitly explain what x is and write an unambiguous statement of $P(x)$.

(ii) Next, write your base case(s) in the box below.

(iii) Finally, provide your induction step. This step will be marked on how clear and mathematically precise your proof is. Ambiguous explanations will have points deducted.
Be sure to clearly describe your induction goal. If you run out of space, continue the proof on the next page (with a note stating that you are writing on the back).

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Question 3 (c, 5): (XC)

We now say that a word w contains a *tripled letter* if some character in the word appears three in a row. For example $w_1 = abbaaabaa$ contains the tripled letter aaa, while $w_2 = abbaabbabbaa$ does not contain any tripled letter.

For a fixed alphabet Σ we let language NTL (no tripled-letter) be the set of all words in Σ^* that do not contain a tripled letter.

Now set $\Sigma = \{a, b\}$. Then

$$NTL = \{\lambda, a, b, aa, ab, ba, bb, baa, aba, bba, abb, bab, aab \ldots\}.$$

Furthermore, for all positive naturals n, define

$$A_n = \{w : \ w \in NTL, \, |w| = n, \text{ and } w \text{ ends in } a\} \quad \text{and} \quad B_n = \{w : \ w \in NTL, \, |w| = n, \text{ and } w \text{ ends in } b\}.$$

As examples, $A_2 = \{aa, ba\}$, $B_2 = \{ab, bb\}$, $A_3 = \{baa, aba, bba\}$, $B_3 = \{abb, bab, aab\}$.

Recall the Fibonacci numbers F_n are defined on the naturals n by: $F_0 = 0$, $F_1 = 1$, and,

$$\forall n \geq 1, \, F_{n+1} = F_n + F_{n-1}. \tag{1}$$

The next four Fibonacci numbers are then $F_2 = 1$, $F_3 = 2$, $F_4 = 3$ and $F_5 = 5$.

Prove by induction that, for all positive naturals, $|A_n| = |B_n| = F_{n+1}$.

(i) First write your induction hypothesis in the box below. This should be in the form $P(x)$, where you *must* explicitly explain what x is and write an unambiguous statement of $P(x)$.

(ii) Next, write your base case(s) in the box below.

(iii) Finally, provide your induction step. This step will be marked on how clear and mathematically precise your proof is. Ambiguous explanations will have points deducted.
Be sure to clearly describe your induction goal. If you run out of space, continue the proof on the next page (with a note stating that you are writing on the back).

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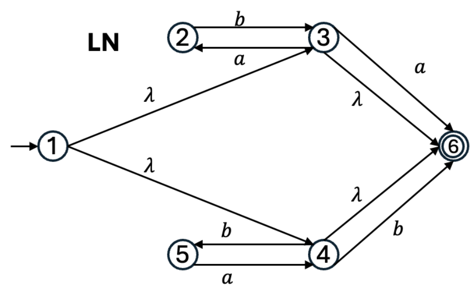
Question 4 (40+5 points total):

This question involves several of the constructions from Kleene's Theorem. We are actually using the language NDL as in Question 3(A). What this means is that you may have extra chances to confirm that your new objects have the same language as before.

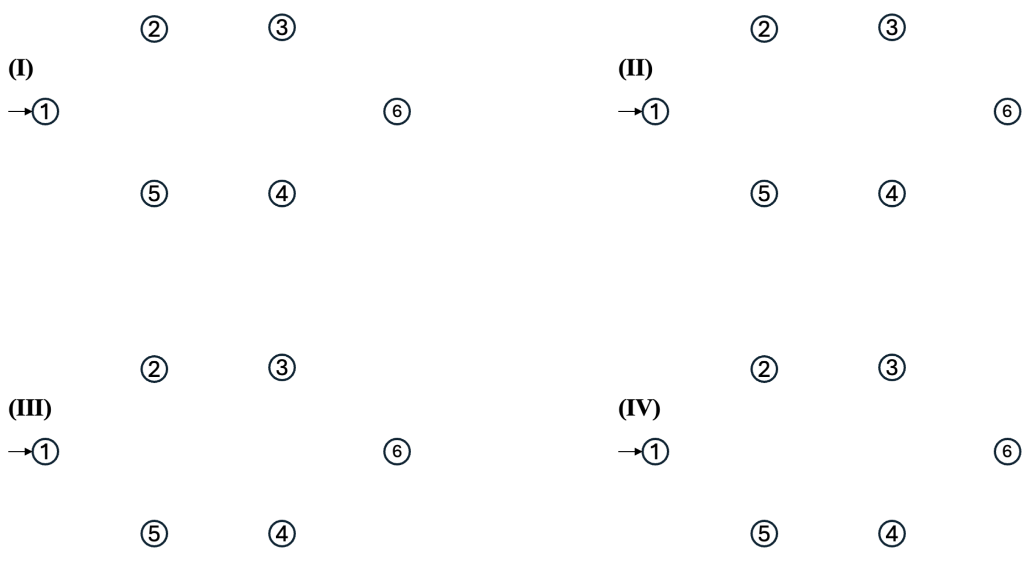
We begin with an ordinary λ-NFA LN, with alphabet $\Sigma = \{a, b\}$, state set $\{1, 2, 3, 4, 5, 6\}$, start state 1, final state set $\{6\}$, and transition relation (with ten total transitions):

$$\{(1, \lambda, 3), (1, \lambda, 4), (2, b, 3), (3, a, 2), (3, a, 6), (3, \lambda, 6), (4, b, 5), (4, b, 6), (4, \lambda, 6), (5, a, 4)\}.$$



(a, 10) Killing λ-moves: *Using the construction* from the lectures and the textbook, find an ordinary NFA ON that is equivalent to the λ-NFA LN.

- In Graph (I), draw all the edges that are contained in the transitive closure of the λ-edges (including the original λ-edges). Label them with λ.
- In Graph (II), draw all the a-letter moves for ON. Label them as a.
- In Graph (III), draw all the b-letter moves for ON. Label them as b.
- In Graph (IV) draw all the edges in ON properly labeled. Also properly denote the final states of ON.



(b, 10) Subset Construction: *Using the Subset Construction*, find a DFA D that is equivalent to the NFA ON. To avoid cascading errors in later sections, *we are going to give you* a DFA below (in the statement for Question 4(d)) that will be equivalent to D. You will be graded on whether you carry out the construction on ON to create your own DFA. (Thus you should name your new states according to the construction.) If you are using the construction correctly, you don't need to explain how you get each step of it.

(c, 10) Minimization: Find a DFA MD that is minimal and has the same language as your DFA D from part (b). If you do not use the Minimization Construction, then prove that your new DFA is minimal and that $L(MD) = L(D)$.

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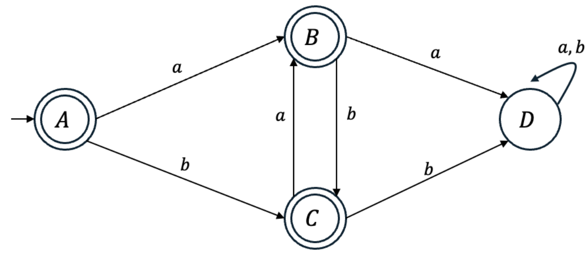
(d, 10) State Elimination: Find and justify a regular expression that is equivalent to the four-state DFA given here, which will be isomorphic to your minimized DFA MD if you solved the earlier parts correctly:



State set $\{A, B, C, D\}$, start state A, final state set $\{A, B, C\}$, transitions $\delta(A, a) = B$, $\delta(A, b) = C$, $\delta(B, a) = D$. $\delta(B, b) = C$, $\delta(C, a) = B$, $\delta(C, b) = \delta(D, a) = \delta(D, b) = D$.

If you use State Elimination on this DFA, no further correctness proof is required. If you use another method, prove that your regular expression is equivalent.

(e, 5XC) Making a λ-NFA: *Using the construction* from lecture and the textbook, compute a λ-NFA from the regular expression in part (d). A different equivalent λ-NFA will get only partial credit. (Though you may omit redundant λ-moves in the construction if you are *sure* that you are not changing the language.)

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Question 5 (20): The following are ten true/false questions, with no explanation needed or wanted, no partial credit for wrong answers, and no penalty for guessing.

After reading the questions, write the correct answer, either T (for true) or F (for false), in the corresponding column. Be sure that your "T" and "F" characters are consistent and distinct.

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)

- (a) The six statements in the dog proof (Question 1) together provide enough information to determine whether Blaze ate Pie.

- (b) Let "likes" be a symmetric binary relation on dogs. Then it is possible that every dog likes at least one other dog, but there is no dog that likes all other dogs.

- (c) Let Q and R each be equivalence relations on the same set S. Assume for elements a, b, c, and d in S, the pairs (a, b) and (c, d) are in Q, and the pair (b, c) is in R. Then it is possible that (a, d) is not in Q.

- (d) Consider the equivalence classes of the integers taken modulo 11. Then it is not true that every class has a multiplicative inverse.

- (e) If a and b are distinct nodes of a directed acyclic graph G, and there is a path in G from a to b, then there is no path in G from b to a.

- (f) Consider a finite game between two players A and B, represented by a game tree with value v. Then whatever strategy A uses, and whatever strategy B uses, the reward at the end of the game will always be v.

- (g) Let Z be the set of all strings w over $\Sigma = \{a, b\}$ such that the number of b's in w is either divisible by 2 or divisible by 3, or both. Then the language of the regular expression $(a^*ba^*ba^*)^* + (a^*ba^*ba^*ba^*)^*$ is exactly Z.

- (h) Let $\Sigma = \{a, b\}$ and let X be the language Σ^*a. Then the languages X and X^R have the same number of Myhill-Nerode equivalence classes. (Recall that X^R is the set of strings whose reversals are in X.)

- (i) There exists a language that can be decided by a Turing machine, but is not the language of any two-way deterministic finite automaton.

- (j) Let X be any language. Then there exists a three-tape nondeterministic Turing machine M such that $X = L(M)$ if and only if X is Turing recognizable.

COMPSCI 250 Final Exam Supplementary Handout: 11 December 2025

At Thanksgiving dinner, there were nine members of Dave's extended family. Dave and his wife Jessica were joined by his daughter Julia, son-in-law Will, grandson Graham, granddaughter Genevieve, his own dogs Blaze and Rhonda, and his daughter's dog Gwen.

Let $F = \{B, D, Ge, Gr, Gw, Je, Ju, R, W\}$ be the family members, and let $D = \{B, Gw, R\}$ be a subset of F containing the dogs. The menu included Dog Food, Green Bean Casserole, Pie, and Turkey — we will consider the set $M = \{DF, GBC, P, T\}$ of those items. Let $A \subseteq F \times M$ be a binary predicate so that $A(x, y)$ means "family member x ate menu item y".

We have six statements about the relation A:

(a, 10) Translations: Translate each statement as indicated.

- **Statement I:** (2, to symbols) Exactly one family member ate all four menu items.
- **Statement II:** (1, to English) $A(Gr, GBC) \rightarrow (A(R, GBC) \wedge A(W, GBC))$.
- **Statement III:** (1, to symbols) If either Will did not eat GBC or Rhonda ate GBC, then Graham ate GBC and Rhonda did not eat GBC.
- **Statement IV:** (2, to English) $\forall x : \forall y : \forall z : [(x \in D) \wedge (y \in D)] \rightarrow (A(x, z) \leftrightarrow A(y, z))$.
- **Statement V:** (2, to symbols) Dave, Julia, and Will each ate exactly the same menu items.
- **Statement VI:** (2, to English) $\forall x : A(x, T) \leftrightarrow (x \neq Je)$.

Here is the λ-NFA from the start of Question 4.

It has alphabet $\Sigma = \{a, b\}$, state set $\{1, 2, 3, 4, 5, 6\}$, start state 1, final state set $\{6\}$, and transition relation (with ten total transitions):

$$\{(1, \lambda, 3), (1, \lambda, 4), (2, b, 3), (3, a, 2), (3, a, 6), (3, \lambda, 6), (4, b, 5), (4, b, 6), (4, \lambda, 6), (5, a, 4)\}.$$