NAME: ______________________________

SPIRE ID: ______________________________

COMPSCI 250
Introduction to Computation
Final Exam Fall 2024

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DIRECTIONS:

- Answer the problems on the exam pages.
- There are **five** problems on pages **2-13**, some with multiple parts, for 125 total points plus 10 extra credit. Final scale will be determined after the exam.
- Page **14** contains useful definitions and is given to you separately – do not put answers on it!
- If you need extra space use the back of a page – both sides are scanned.
- But, if you do write a solution on the back, you must explicitly have a note on the front stating that you used the back page.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer.
- Your answers must be LEGIBLE, and not cramped. Write only short paragraphs with space between paragraphs.

1	**/35**
2	**/10+10**
3	**/20**
4	**/40**
5	**/20**
Total	**/125+10**

Family Name: ____________________

Question 1 (Dog Proof, 35 total points) This fall, Dave's family (Blaze, Dave, Jessica, and Rhonda) had a number of notable encounters with wildlife. Let M be the set $\{b, d, j, r\}$ of the four Members in the family, and let A be the set $\{B, D, F, O, W\}$ of the Animals (Bear, Deer, Fox, Opossum, and Wolverine) that they may have observed. Let $S(x, y)$ be the predicate meaning "Member x saw Animal y". Finally, H is the subset $\{d, j\}$ of the Members that are Human, and $H(x)$ is the corresponding unary predicate on the set M meaning "Member x is Human".

(A, 10) Translations: Translate each of these statements as indicated (two points each).

- **Statement I** (to English): $(S(b, F) \vee S(d, B)) \rightarrow (\neg S(d, B) \wedge S(d, O))$

- **Statement II** (to symbols): Dave saw an Opossum, and either Blaze saw a Fox or Dave saw a Bear, but not both.

- **Statement III** (to English): $\forall a : (\exists u : S(u, a)) \rightarrow (\exists v : H(v) \wedge S(v, a))$

- **Statement IV** (to symbols): Blaze and Jessica each saw exactly one animal.

- **Statement V** (to English): $\forall x : \exists y : \exists z : (y \neq z) \wedge \neg S(x, y) \wedge \neg S(x, z)$

(B, 10) Boolean Proof: Using only Statements I and II, let $p = S(b, F)$, $q = S(d, B)$, and $r = S(d, O)$. Prove that p and r are true, and that q is false. You may use either a truth table or deductive rules. If you use deductive rules, remember that you must prove *both* that this solution agrees with the statements, and that no other other solution agrees with the statements.

(C, 15) Predicate Proof: In this problem you must determine all twenty boolean values of the predicate $S(x, y)$ for every member x and every animal y. You may use Statements I, II, III, IV, and V, as well as:

Statement VI: Rhonda saw a Bear, a Deer, and a Fox.

Do this by filling out this table, with a "0" for "false" and a "1" for "true".

$S(b, B) =$		$S(d, B) =$ **0**		$S(j, B) =$		$S(r, B) =$
$S(b, D) =$		$S(d, D) =$		$S(j, D) =$		$S(r, D) =$
$S(b, F) =$ **1**		$S(d, F) =$		$S(j, F) =$		$S(r, F) =$
$S(b, O) =$		$S(d, O) =$ **1**		$S(j, O) =$		$S(r, O) =$
$S(b, W) =$		$S(d, W) =$		$S(j, W) =$		$S(r, W) =$

You must **justify** each of the twenty answers in this table, except for the three that have been filled for you. (These are the answers from Part (B).) Many of these answers have very short justifications, but some are more involved and may require quoting other answers you have already derived. If your answer involves the use of a quantifier proof rule, and many of them should, make clear which rule you are using and when. Using complete sentences will generally make your answers more readable.

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Question 2 (A) (10): (Induction 1)

For $n \geq 1$, set $S(n) = \sum_{i=1}^{n}(4i - 1)$.

Note that $S(3) = 3 + 7 + 11 = 21 = 3 \cdot 7 = 3 \cdot (2 \cdot 3 + 1)$ and

$S(4) = 3 + 7 + 11 + 15 = 36 = 4 \cdot 9 = 4 \cdot (2 \cdot 4 + 1)$.

Prove by induction that, for all positive naturals n, $S(n) = n(2n + 1)$.

(i) First write your induction hypothesis in the box below. This should be in the form $P(x)$, where you *must* explicitly explain what x is and write an unambiguous statement of $P(x)$.

(ii) Next, write your base case(s) in the box below.

(iii) Finally, provide your induction step. This step will be marked on how clear and mathematically precise your proof is. Ambiguous explanations will have points deducted.
Be sure to clearly describe your induction goal. If you run out of space, continue the proof on the back page (with a note stating that you are writing on the back).

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Question 2 (B) (10): (Induction Extra Credit)

For $n \geq 0$, set $S(n) = \sum_{i=0}^{n} i2^i$.

Note that $S(1) = 1{\cdot}2^1 = 2 = 2(1{\cdot}2^1{-}2^1{+}1)$ and $S(2) = 1{\cdot}2^1{+}2{\cdot}2^2 = 10 = 2(2{\cdot}2^2{-}2^2{+}1)$.

Prove by induction that, for all naturals n, $S(n) = 2\,(n2^n - 2^n + 1)$.

(i) First write your induction hypothesis in the box below. This should be in the form $P(x)$, where you *must* explicitly explain what x is and write an unambiguous statement of $P(x)$.

(ii) Next, write your base case(s) in the box below.

(iii) Finally, provide your induction step. This step will be marked on how clear and mathematically precise your proof is. Ambiguous explanations will have points deducted.

Be sure to clearly describe your induction goal. If you run out of space, continue the proof on the back page (with a note stating that you are writing on the back).

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Question 3 (20 points): (Induction 2) Define the language $\mathbf{B}$ on $\Sigma = \{a, b, c\}$ as follows.

String $\mathbf{w} \in \Sigma^*$ is in $\mathbf{B}$ if

R1: $\mathbf{w} = \lambda$ (the empty string) or
R2: $\mathbf{w} = a\mathbf{v}b$ where $\mathbf{v} \in \mathbf{B}$ or
R3: $\mathbf{w} = a\mathbf{u}b\mathbf{v}c$ where $\mathbf{u} \in \mathbf{B}$ and $\mathbf{v} \in \mathbf{B}$.

No other strings are in $\mathbf{B}$.

Define $N_a(\mathbf{w})$, $N_b(\mathbf{w})$, and $N_c(\mathbf{w})$, to be, respectively, the number of a's, b's and c's in $\mathbf{w}$.

Also define $|\mathbf{w}| = N_a(\mathbf{w}) + N_b(\mathbf{w}) + N_c(\mathbf{w})$ to be the total number of characters in $\mathbf{w}$.

Examples: Let $\mathbf{w}_1 = ab$, $\mathbf{w}_2 = abc$, $\mathbf{w}_3 = aaabbbabc$, $\mathbf{w}_4 = aba$, $\mathbf{w}_5 = abcc$.
Then $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathbf{B}$, $\mathbf{w}_4, \mathbf{w}_5 \notin \mathbf{B}$
$N_a(\mathbf{w}_1) = N_b(\mathbf{w}_1) = 1, N_c(\mathbf{w}_1) = 0.$ $\quad N_a(\mathbf{w}_2) = N_b(\mathbf{w}_2) = N_c(\mathbf{w}_2) = 1.$
$N_a(\mathbf{w}_3) = N_b(\mathbf{w}_3) = 4,\ N_c(\mathbf{w}_3) = 1.$

A *derivation* that $\mathbf{w} \in \mathbf{B}$ is a listing of the rules that show that $\mathbf{w} \in \mathbf{B}$, one item per line. Each line in the derivation should justify the creation of a specific string in $\mathbf{B}$. The final line in the derivation should be the justification of $\mathbf{w}$.

Here is a derivation that $aaabbabcb \in \mathbf{B}$.
1. $\mathbf{v_1} = \lambda \in \mathbf{B}$. (R1)
2. $\mathbf{v_2} = ab = a\mathbf{v_1}b \in \mathbf{B}$. (R2 and line 1)
3. $\mathbf{v_3} = a\,ab\,b\,ab\,c = a\mathbf{v_2}b\mathbf{v_2}c \in \mathbf{B}$. (R3 and line 2)
4. $\mathbf{w} = a\,aabbabc\,b = a\mathbf{v_3}b \in \mathbf{B}$. (R2 and line 3)

Part (A) of this problem is below. Parts (B) and (C) are on the following pages. When writing their solutions, you must use the the mathematical notation we provided above.

If your proofs of (B) and (C) have multiple pieces, place each piece in a separate paragraph with space between the paragraphs. Be sure to clearly describe your induction goal.

If you need more space, continue your proof on the back of the page, noting explicitly that you are continuing on the back.

(A) (2 pts) Give a derivation that shows that the string $\mathbf{w} = aabcbaabbc$ is in $\mathbf{B}$.
Write your answers on the lines below (you shouldn't need all of the lines.).

1. ____________________
2. ____________________
3. ____________________
4. ____________________
5. ____________________
6. ____________________
7. ____________________

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(B) 9 points. Prove by induction that, for every $\mathbf{w} \in B$, $N_a(\mathbf{w}) = N_b(\mathbf{w})$.

a) First write your induction hypothesis in the box below. This should be in the form $P(x)$, where you *must* explicitly explain what x is and write an unambiguous statement of $P(x)$.

(b) Next, write your base case(s) in the box below.

(c) Finally, provide your induction step. This step will be marked on how clear and mathematically precise your proof is. Ambiguous explanations or explanations missing details will have points deducted.

(C) 9 points. Prove by induction that, for every $\mathbf{w} \in B$, $N_c(\mathbf{w}) \leq N_a(\mathbf{w})$.

a) First write your induction hypothesis in the box below. This should be in the form $P(x)$, where you *must* explicitly explain what x is and write an unambiguous statement of $P(x)$.

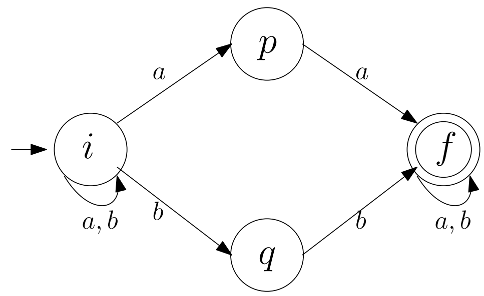
(b) Next, write your base case(s) in the box below.

(c) Finally, provide your induction step. This step will be marked on how clear and mathematically precise your proof is. Ambiguous explanations or explanations missing details will have points deducted.

Question 4 (40 points total):

This question involves several of the constructions from Kleene's Theorem. We begin with an ordinary NFA N, with alphabet $\Sigma = \{a, b\}$, state set $\{i, p, q, f\}$, start state i, final state set $\{f\}$, and transition relation

$$\{(i, a, i), (i, b, i), (i, a, p), (i, b, q), (p, a, f), (q, b, f), (f, a, f), (f, b, f)\}.$$

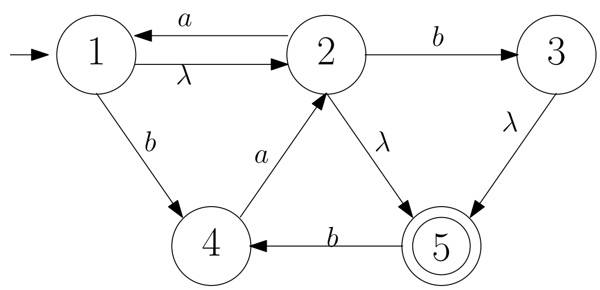


(a, 10) Subset Construction: *Using the Subset Construction*, find a DFA D that is equivalent to the NFA N. It's sufficient to show the new DFA diagram, without further explanation, if it comes from the given construction.

(b, 10) Minimization: Find a DFA D' that is minimal and has the same language as your DFA D from part (a). If you do not use the Minimization Construction, prove that your new DFA is minimal and that $L(D) = L(D')$.

(c, 10) State Elimination: Find and justify a regular expression that is equivalent to your DFA D and your minimized D'. If you use State Elimination on either DFA, no further correctness proof is required. If you use another method, prove that your regular expression is equivalent.

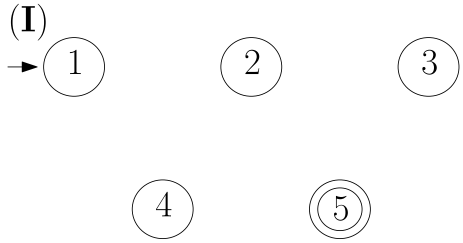
For the last part of this problem, we will work with a *different* language and machine. Here is a new λ-NFA K, which has alphabet $\Sigma = \{a, b\}$, state set $\{1, 2, 3, 4, 5\}$, start state 1, final state set $\{5\}$, and transition relation $(1, \lambda, 2), (1, b, 4), (2, a, 1), (2, b, 3), (2, \lambda, 5), (3, \lambda, 5), (4, a, 2), (5, b, 4)$.



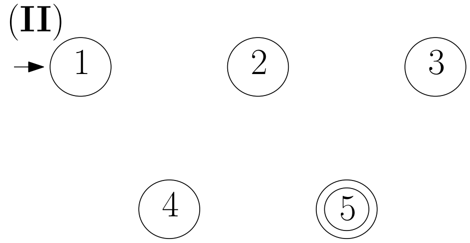
(d, 10) Killing λ-moves: *Using the construction* from the lectures and the textbook, find an ordinary NFA K' that is equivalent to the λ-NFA K above.

- In Graph (I), draw all the edges that are contained in the transitive closure of the λ-edges. Label them with λ.
- In Graph (II), draw all the a-letter moves for K'. Label them as a.
- In Graph (III), draw all the b-letter moves for K'. Label them as b.
- In Graph (IV) draw all the edges in K' properly labeled. Also properly denote the final states of K'.

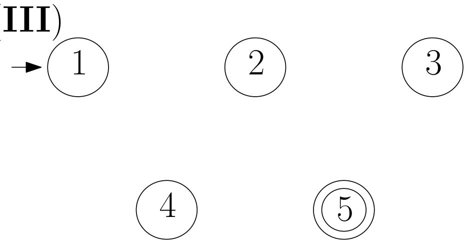
(I)



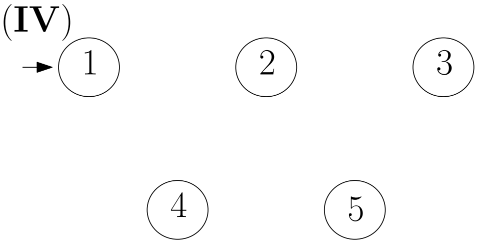
(II)



(III)



(IV)



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Question 5 (20): The following are ten true/false questions, with no explanation needed or wanted, no partial credit for wrong answers, and no penalty for guessing.

After reading the questions, write the correct answer, either T (for true) or F (for false), in the corresponding column. Be sure that your "T" and "F" characters are consistent and distinct.

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)

- (a) A function $f : A \to B$ is one-to-one if and only if for every element x of A, there is exactly one element of B mapped from x.

- (b) Every positive integer is divisible by some prime number.

- (c) Let G be a strongly connected directed graph with finitely many vertices. Let s be one of the nodes of G. Then both a depth-first search and a breadth-first search will each construct a directed tree rooted at s, in which every node of G will appear in each tree.

- (d) Let u and v be any two strings over the same alphabet. Then u is a substring of v if and only if u is both a prefix of v and a suffix of v.

- (e) Let $P(R)$ be a predicate where the variable R ranges over all regular expressions that have alphabet $\Sigma = \{a, b\}$. If we prove $P(\emptyset)$, $P(a)$, $P(b)$, and $\forall R : \forall S : (P(R) \wedge P(S)) \to (P(R{+}S) \wedge P(RS) \wedge P(R^*))$, we may conclude that $P(R)$ is true for all regular expressions over this alphabet.

- (f) Let M be a Turing machine with start state i and blank symbol $\Box$. Then if $\delta(i, \Box) = (i, \Box, L)$, the language $L(M)$ of M is $\{\lambda\}$.

- (g) The two regular expressions $a(a + b)^*b + a(a + b)^*a$ and $aa(a + b)^* + ab(a + b)^*$ represent the same regular language.

- (h) Suppose we conduct both a UCS search and an A^* search on the same weighted directed graph. Then it is possible that the two searches find different paths from the start node to the goal node.

- (i) It is possible for an NFA diagram to also represent a DFA.

- (j) If we apply the Subset Construction to any NFA, the resulting DFA will always have strictly more states than the original NFA had.

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- **Statement I** (to English): $(S(b,F) \vee S(d,B)) \rightarrow (\neg S(d,B) \wedge S(d,O))$

- **Statement II** (to symbols): Dave saw an Opossum, and either Blaze saw a Fox or Dave saw a Bear, but not both.

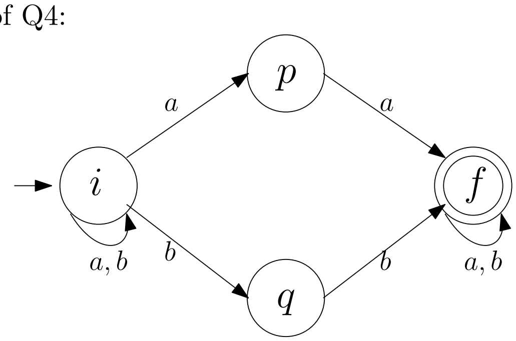
- **Statement III** (to English): $\forall a : (\exists u : S(u,a)) \rightarrow (\exists v : H(v) \wedge S(v,a))$

- **Statement IV** (to symbols): Blaze and Jessica each saw exactly one animal.

- **Statement V** (to English): $\forall x : \exists y : \exists z : (y \neq z) \wedge \neg S(x,y) \wedge \neg S(x,z)$

- **Statement VI:** Rhonda saw a Bear, a Deer, and a Fox.

The NFA N from the start of Q4:



The NFA K from Q4(e):