NAME: ______________________________

SPIRE ID: ______________________________

COMPSCI 250
Introduction to Computation
First Midterm Fall 2024

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DIRECTIONS:

- Answer the problems on the exam pages.
- There are four problems on pages 2-9, some with multiple parts, for 100 total points plus 5 extra credit. Final scale will be determined after the exam.
- Page 10 contains useful definitions and is given to you separately – do not put answers on it!
- If you need extra space use the back of a page – both sides are scanned.
  But, if you do write on the back, you must explicitly have a note on the front stating that you used the back page.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer.
- Your answers must be LEGIBLE, and not cramped. Write only short paragraphs with space between paragraphs

1	/10
2	/10
3	/20
4	/20
5	/20+5
6	/20
Total	/100+5

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Dave's dogs Blaze and Rhonda are each served a meal in the morning, in the afternoon, and in the evening. Each is given their own bowl, but they do not always each eat their own food. In this problem you will determine which dog ate which meal at which time on a particular day.

Let D be the set $\{B, R\}$ of dogs consisting of Blaze and Rhonda. Let T be the set $\{m, a, e\}$ of meal times, consisting of "morning", "afternoon", and "evening". Let $A \subseteq D \times D \times T$ be a relation such that $A(x, y, t)$ means "dog x ate the designated meal for dog y at time t".

Question 1 (10): (Translations) Translate each statement as according to the directions:

- (a, 2) (to symbols) (Statement I)
  Rhonda ate Blaze's food in the morning, and if she did not also eat her own food in the morning, then she did not eat Blaze's food in the morning.

- (b, 2) (to English) (Statement II)
  $A(r, r, a) \rightarrow (A(r, b, m) \wedge \neg A(r, r, m))$

- (c, 2) (to symbols) (Statement III)
  At each time, each dog's meal was eaten by exactly one dog.

- (d, 2) (to English) (Statement IV)
  $\forall t : A(b, b, t) \leftrightarrow (t = e)$

- (e, 2) (to symbols) (Statement V)
  At every meal time, Rhonda ate at least one meal.

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Question 2 (10): (Boolean Proof) Both Questions 2 and 3 use the definitions, predicates, and statements above.

Using *only* Statements I and II, **prove** the truth value of each of the three propositions $p = A(r, b, m)$, $q = A(r, r, m)$, and $r = A(r, r, a)$.

You may use a truth table or a deductive sequence proof. Don't forget that if you use a deductive sequence proof, your argument must show both that your solution satisfies both of the statements, and that there is no other solution.

To keep you from getting off track for Question 3, we're going to tell you that p and q are true, and that r is false. But your answer to *this* question should prove those three assertions, not assume them.

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Question 3 (20) (Predicate Proof) : This question also uses the definitions, predicates, and statements above. Now assume that all of Statements I-V are true. (You may quote the results of Question 2, given above, whether or not you proved them.)

Determine all twelve truth values for the relation A. That is, for all dogs x, for all dogs y, and all times t, determine the value of $A(x, y, t)$. To show your answer, for each of the 12 propositions, you should circle either True or False appropriately in the table below.

Each entry in the table needs to be justified, either from the premises or from answers that you have already determined. Since many of the premises involve quantifiers, your justifications will involve quantifier proofs as are typical in dog proofs in this course. Your justifications may use either English or symbolic notation, or a combination of both, but it should be clear to the reader when you are using each of the quantifier rules.

$A(b, b, m)$	True / False
$A(r, b, m)$	True / False
$A(r, r, m)$	True / False
$A(b, r, m)$	True / False
$A(b, b, a)$	True / False
$A(r, b, a)$	True / False
$A(r, r, a)$	True / False
$A(b, r, a)$	True / False
$A(b, b, e)$	True / False
$A(r, b, e)$	True / False
$A(r, r, e)$	True / False
$A(b, r, e)$	True / False

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Question 4 (20): (Binary Relations on a Set) Parts (a) and (b) deal with two binary relations P and Q, each from the same set $A = \{a, b, c, d\}$ to itself. Part (a) is on this page, part (b) on the next.

(a, 10) The relation $P \subseteq A \times A$ is a **partial order**.
It is known that $(b, c) \in P$, $(d, c) \in P$, $(c, a) \in P$ and $(a, c) \notin P$.

(i) In the table given below label all pairs that must be in P with a "✓" and all pairs that cannot be in P with a "×." For pairs that are neither, leave their entry blank. We start you off by labelling the four known pairs with the appropriate "✓" or ×.

Justify your answers. That is, you must provide a brief explanation as to why each of the pairs you marked "✓" are in P and those you marked "×" are not in P.

Your explanations must reference the properties of P that are being used. They must also be consistent. That is, if you somewhere use the fact that $(x, y) \notin P$ where $(x, y) \neq (a, c)$, then you must have already previously proven that $(x, y) \notin P$.

(ii) How many partial orders P exist that satisfy $(b, c) \in P$, $(d, c) \in P$, $(c, a) \in P$ and $(a, c) \notin P$? Briefly explain how you know this.

(a,a)	(a,b)	(a,c)	(a,d)	(b,a)	(b,b)	(b,c)	(b,d)	(c,a)	(c,b)	(c,c)	(c,d)	(d,a)	(d,b)	(d,c)	(d,d)
		×				✓		✓						✓	

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(b, 10) The relation $Q \subseteq A \times A$ is an **equivalence relation**.
It is known that $(a, c) \in Q$, $(b, c) \in Q$ and $(a, d) \notin Q$.

(i) In the table given below, label all pairs that must be in Q with a "✓" and all pairs that cannot be in Q with a "×." For pairs that are neither, leave their entry blank. We start you off by labelling the three known pairs with the appropriate "✓" or ×.

Justify your answers following the same rules that were given in part (a).

(ii) How many partial orders Q exist that satisfy $(a, c) \in Q$, $(b, c) \in Q$, and $(a, d) \notin Q$? Briefly explain how you know this.

(a,a)	(a,b)	(a,c)	(a,d)	(b,a)	(b,b)	(b,c)	(b,d)	(c,a)	(c,b)	(c,c)	(c,d)	(d,a)	(d,b)	(d,c)	(d,d)
		✓	×			✓									

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Question 5 (20+5): (Number Theory)

- (a, 4) In the box below, write down the complete statement of the Chinese Remainder Theorem (for two moduli) as it was taught in class. Be careful to explicitly state the conditions required to be able to use it.

- (b, 8) The naturals 102 and 23 are relatively prime.
  Use the Extended GCD algorithm as taught in class to find integers a and b satisfying the equation $102 \cdot a + 23 \cdot b = 1$. Show all of your work.

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- (c, 4) Using the results from the previous part, determine both an inverse of 23 modulo 102 and an inverse of 102 modulo 23.
  For full credit, your answers should each be the smallest natural numbers that are inverses.

- (d, 4) Using the results from the previous parts, determine the smallest natural x that solves both the congruences

$$x \equiv 2 \pmod{23} \quad \text{and} \quad x \equiv 1 \pmod{102}.$$

- (e, 5 extra credit) Find the smallest natural y that satisfies both the congruences in part (d) and also has a decimal representation ending in 9 (that is, $y \equiv 9 \pmod{10}$). Remember that you may give the result as an arithmetic expression, as long as you justify it.

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Question 6 (20): The following are ten true/false questions, with no explanation needed or wanted, no partial credit for wrong answers, and no penalty for guessing.

After reading the questions, write the correct answer, either T (for true) or F (for false), in the corresponding column.

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)

- (a) If A is a finite set and R is a relation from A to A, and it is an injection (a one-to-one function), then R is a bijection.

- (b) If R is a reflexive and transitive relation from A to A and R is not antisymmetric, then R must be symmetric.

- (c) Recall that "u^R" is the reversal of a string u. If u and v are any strings, then the reversal of the string uvu^R is uv^Ru^R.

- (d) The Sieve of Eratosthenes is used to solve a pair of congruences with different prime moduli.

- (e) Let X and Y be any finite sets. Then $|X \setminus Y| \leq |Y|$.

- (f) Suppose we are trying to prove a compound proposition X (where X involves boolean variables p and q plus possibly others), and our premise is $p \to q$. Then it is *insufficient* to prove the two cases $(p \wedge q) \to X$ and $\neg p \to X$.

- (g) For all i in the set $S = \{1, 2, 3, 4\}$, let p_i and q_i each be prime numbers. If $p_1p_2p_3p_4 = q_1q_2q_3q_4$, then the statement $\forall i : \exists j : p_i = q_j$ is true, where the variables range over S.

- (h) Let A be a finite set. Then the empty string λ is a member of the language A^* if and only if A is non-empty.

- (i) There exists a positive natural m (*i.e.*, with $m \geq 1$) such that every prime number falls into the same congruence class modulo m.

- (j) Let R be a relation from a finite set A to a finite set B. Assume that for every $a \in A$, there exists an element $b \in B$ such that $(a, b) \in R$. Then R is a function if and only if for every $b \in B$, there exists an element $a \in A$ such that $(a, b) \in R$.

COMPSCI 250 First Midterm Supplementary Handout: 10 October 2024

Here are definitions of sets, predicates, and statements used on the exam.

Remember that the scope of any quantifier is always to the end of the statement it is in.

The scenario of Question 1-3 is as follows.

Dave's dogs Blaze and Rhonda are each served a meal in the morning, in the afternoon, and in the evening. Each is given their own bowl, but they do not always each eat their own food. In this problem you will determine which dog ate which meal at which time on a particular day.

Let D be the set $\{B, R\}$ of dogs consisting of Blaze and Rhonda. Let T be the set $\{m, a, e\}$ of meal times, consisting of "morning", "afternoon", and "evening". Let $A \subseteq D \times D \times T$ be a relation such that $A(x, y, t)$ means "dog x ate the designated meal for dog y at time t".

The five statements of Question 1 are:

- (a, 2) (to symbols) (Statement I)

  Rhonda ate Blaze's food in the morning, and if she did not also eat her own food in the morning, then she did not eat Blaze's food in the morning.

- (b, 2) (to English) (Statement II)

  $A(r, r, a) \rightarrow (A(r, b, m) \wedge \neg A(r, r, m))$

- (c, 2) (to symbols) (Statement III)

  At each time, each dog's meal was eaten by exactly one dog.

- (d, 2) (to English) (Statement IV)

  $\forall t : (A(b, b, t) \leftrightarrow (t = e)$

- (e, 2) (to symbols) (Statement V)

  At every meal time, Rhonda ate at least one meal.