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COMPSCI 250
Introduction to Computation
First Midterm Fall 2023

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DIRECTIONS:

- Answer the problems on the exam pages.
- There are four problems on pages 2-8, some with multiple parts, for 100 total points plus 5 extra credit. Final scale will be determined after the exam.
- Page 9 contains useful definitions and is given to you separately – do not put answers on it!
- If you need extra space use the back of a page.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer.

1	/10
2	/10
3	/20
4	/20
5	/20+5
6	/20
Total	/100+5

This problem deals with a set D of exactly eight distinct dogs named August (a), Blaze (b), Bliss (b'), Daisy (d), Ollie (o), Rhonda (r), Scout (s), and Tuesday (t). Note that Blaze and Bliss have similar names but have different abbreviations.

Each dog has one of the following jobs, from the set $J = \{FD, HP, SD\}$, as a Farm Dog, a House Pet, or a Service Dog. The function e from D to J is defined so that for any dog x in D, $e(x)$ is the job of x.

You are told some of the values of the function e. Blaze and Rhonda are Dave's dogs and are both House Pets. Ollie, who attends lecture -01 of COMPSCI 250 this term, is a Service Dog. And Scout, who has appeared in prior 250 exams, is a Farm Dog.

The binary relation P on the set D is defined so that for any dogs x and y, $P(x, y)$ is true if and only x's name *precedes* y's name in alphabetical order. So, for example, $P(b, b')$ is true and both $P(b, b)$ and $P(b', b)$ are false. You may assume that the relation P is antireflexive, antisymmetric, and transitive.

Question 1 (10): (Translations) Translate each statement as according to the directions:

- (a, 2) (to English) (Statement I) $(e(a) = SD) \rightarrow (e(b') = SD) \wedge (e(d) = SD))$

- (b, 2) (to symbols) (Statement II) If Bliss is a Service Dog, then so is either (or both) of August and Daisy.

- (c, 2) (to English) (Statement III) $((\neg(e(a) = SD) \vee (e(b') = SD)) \rightarrow \neg(e(d) = SD))$

- (d, 2) (to symbols) (Statement IV) For every dog x, there is a dog different from x with the same job as x.

- (e, 2) (to English) (Statement V) $\forall x : \forall y : ((e(x) = e(y)) \wedge P(x, y)) \rightarrow [\exists z : P(x, z) \wedge P(z, y)]$

Question 2 (10): (Boolean Proof) Questions 2 and 3 use the definitions, predicates, and statements above.

Using *only* Statements I, II, and III, determine the truth values of the three statements p defined as $e(a) = SD$, q defined as $e(b') = SD$, and r defined as $e(d) = SD$. It may be easier to refer to the three atomic propositions as p, q, and r. You may use a truth table or a deductive sequence proof.

Question 3 (20) (Predicate Proof) : This question also uses the definitions, predicates, and statements above. Now assume that all of Statements I-V are true, and assume the four values of the function e provided above. Using propositional and quantifier rules, determine the jobs of the four remaining dogs, that is, the function values $e(a)$, $e(b')$, $e(d)$, and $e(t)$.

Question 4 (20): (Number Theory) This problem concerns the naturals 55 and 78.

Here are your questions:

- (a, 3) Write each of the naturals 55 and 78 as a product of prime numbers.

- (b, 3) Explain why the result of part (a) implies that the naturals 55 and 78 are relatively prime.

- (c, 4) Carry out the ordinary Euclidean Algorithm on 55 and 78, and explain why the result tells you that 55 and 78 are relatively prime.

- (d, 10) Justifying your answers, find an inverse of 55, modulo 78, and an inverse of 78, modulo 55. Make sure that you indicate which is which.

Question 5 (20+5): (Linear Functions) Let n be a positive natural, and let S be set of naturals $\{0, 1, \ldots, n-1\}$.

Given any two elements a and b of S, define a function f_{ab} from S to S such that $f_{ab}(x) = (ax + b)\%n$, using the Java or Python notation % for the remainder operator. (For any naturals x and y, $x\%y$ is defined to be the remainder when x is divided by y, so that for example $11\%3 = 2$.)

- (a, 5) Prove that if a, b, c, and d are any elements of S, the function composition $f_{cd} \circ f_{ab}$ is another function f_{pq}, for some two element p and q in S. What are the values p and q in this case?

- (b, 5) For a given n, for what values a and b is the function f_{ab} a bijection from S to S?

- (c, 5XC) Let n be a prime number. Let w, x, y, and z be elements of S, and assume $w \neq y$. Prove that there is exactly one pair (a, b) in $S \times S$ such that $f_{ab}(w) = x$ and $f_{ab}(y) = z$.

- (d, 10) For $n = 78$, $a = 55$, and $b = 36$, compute the inverse of the function f_{ab}. That is, find elements c and d of S such that $f_{cd} \circ f_{ab}$ is the identity function on S. You may find some of the calculations from Question 4(d) to be useful.

Question 6 (20): The following are ten true/false questions, with no explanation needed or wanted, no partial credit for wrong answers, and no penalty for guessing.

- (a) Let $P(x)$ be a unary predicate on integers. If $\forall x : (x > 0) \rightarrow P(x)$ and $\forall x : (x < 0) \rightarrow P(x)$ are both true, then we may conclude $\forall x : P(x)$.

- (b) Let A be any non-empty set. Then there exists a binary relation on A that is both an equivalence relation and a partial order.

- (c) Let all variables range over the type T. Then the statement "$\forall x : P(x)$" is equivalent to "$\forall x : (x \in T) \rightarrow P(x)$", and the statement "$\exists x : P(x)$" is equivalent to "$\exists x : (x \in T) \wedge P(x)$".

- (d) Let T be a type including several dogs, one of whom is Rhonda (r). If we are given the assumption $\exists x : P(x)$, where the variable x is over T, then we may conclude $P(r)$ by Instantiation.

- (e) Let a, b, c, and d be any four *integers* (positive, negative, or zero). If $\exists c : ac = b$ and $\exists d : bd = a$ are both true, then $a = b$ must be true.

- (f) The Contrapositive Rule implies that the negation of the statement "$p \rightarrow q$" is equivalent to "$q \rightarrow p$".

- (g) Let n be any positive natural and let A be a language. Then there exists a string in A^* with more than n letters if and only if there exists a non-empty string (a string other than λ) in A.

- (h) Suppose we want to prove "All corgis are both lazy and silly". We could do this by making the assumption "There exists a corgi that is both lazy and silly", then deriving a contradiction from this assumption.

- (i) Let A be an non-empty set, let x be an element of A, and let R be a binary relation on A such that R is a subset of the relation $\{(x, y) : y \in A\}$, that is, R is a subset of the set of pairs whose first member is always x. Then R may fail to be a transitive relation.

- (j) There exists a set A of naturals such that A is the set of naturals that are both even and odd.

COMPSCI 250 First Midterm Supplementary Handout: 12 October 2023

Here are definitions of sets, predicates, and statements used on the exam.

Remember that the scope of any quantifier is always to the end of the statement it is in.

In Question 4 we often use the Java operator % on naturals, so that $x\%y$ is the remainder when x is divided by y.

The five statements of Question 1 are:

- (a, 2) (to English) (Statement I) $(e(a) = SD) \rightarrow (e(b') = SD) \wedge (e(d) = SD))$

- (b, 2) (to symbols) (Statement II) If Bliss is a Service Dog, then so is either (or both) of August and Daisy.

- (c, 2) (to English) (Statement III) $((\neg(e(a) = SD) \vee (e(b') = SD)) \rightarrow \neg(e(d) = SD))$

- (d, 3) (to symbols) (Statement IV) For every dog x, there is a dog different from x with the same job as x.

- (e, 3) (to English) (Statement V) $\forall x : \forall y : ((e(x) = e(y)) \wedge P(x, y)) \rightarrow [\exists z : P(x, z) \wedge P(z, y)]$

You are also told (for Question 3) that Blaze and Rhonda are House Pets, Ollie is a Service Dog, and Scout is a Farm Dog. The entire set of dogs, in alphabetical order, are August, Blaze, Bliss, Daisy, Ollie, Rhonda, Scout, and Tuesday.