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COMPSCI 250
Introduction to Computation
Final Exam Fall 2021

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14 December 2021

DIRECTIONS:

- Answer the problems on the exam pages.
- There are four problems on pages 2-8, some with multiple parts, for 125 total points plus 10 extra credit points. Probable scale is somewhere around A=105, C=70, but will be determined after we grade the exam.
- Page 9 contains useful definitions for Question 1 and is given to you separately – do not put answers on it!
- If you need extra space use the back of a page.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer.

1	/30
2	/30+10
3	/35
4	/30
Total	/125+10

Question 1 (30): Five of the dogs currently on Dave's block form the set $D = \{b, c, i, k, m\}$, consisting of Blaze, Clover, Indy, Kiké, and Mango. We define three predicates on the set D, such that $R(x)$ means "dog x is a retriever", $S(x, y)$ means "dog x is smaller than dog y", and $L(x, y)$ means "dog x is larger than dog y".

We assume that both the relations L and S are each antireflexive, antisymmetric, and transitive. We also assume that $\forall x : \forall y : L(x, y) \leftrightarrow S(y, x)$, and that no two dogs have exactly the same size (in symbols, $\forall x : \forall y : (x \neq y) \rightarrow (S(x, y)) \vee S(y, x)))$.

We also define two more predicates on D in terms of the first two, so that $LR(x)$ means "dog x is larger than some retriever" and $SR(x)$ means "dog x is smaller than some retriever".

(a, 10): Translate the following statements as indicated.

- Statement I: (into symbols, using only quantifiers, R, and S) Mango, who is not a retriever, is not smaller than any retriever.

- Statement II: (into English) $\forall x : SR(x) \oplus LR(x)$.

- Statement III: (into English) $\exists x : L(b, x) \wedge \neg R(x)$.

- Statement IV: (into symbols) Both Blaze and Kiké are each smaller than every retriever.

- Statement V: (into English) $(S(b, c) \leftrightarrow S(b, i)) \wedge (S(b, k) \oplus S(b, m))$

(b, 10): Using any or all of Statements I-V, determine Blaze's order relative to the other four dogs, by finding the truth values of $S(b,c)$, $S(b,k)$, $S(b,i)$, and $S(b,m)$. (**Note:** This is a bit different from the boolean question on prior exams, as there is only one purely boolean statement. You will get half credit for correctly determining (by a truth table or otherwise) which settings of the truth values make Statement V true. You will need at least some of the other four statements to fully answer the question. You could also fully solve it without Statement V at all.)

(c, 10): Assuming Statement II, and the assumptions and definitions above, determine exactly how many retrievers there are in the set D, and prove your answer.

Question 2 (30): (a, 10) Prove that for any natural n, the sum $\sum_{i=1}^{n} \frac{1}{3^i}$ equals $\frac{1}{2} - \frac{1}{2 \cdot 3^n}$. Don't forget to handle the case of $n = 0$, because 0 is a natural. Remember that sums with no terms always equal 0.

CICS planned a big banquet to celebrate some large donations, and they planned to seat each of $2n$ guests at a long table, with the seats $\{1, 2, \ldots, 2n\}$ in a single row. But with the advent of the Omicron COVID variant, they had to reduce the guest list, even though there are still $n + 1$ important guests that still need to be invited.

(b, 10) The first guidelines say that no two guests may sit in adjacent seats (such as i and $i + 1$). Prove by induction, for any positive n, that we cannot seat $n + 1$ people among positions $\{1, 2, \ldots 2n\}$ without using at least one pair of adjacent seats. Prove this by ordinary induction.

CICS asked for new guidelines and were told now that seats could be adjacent, as long as they don't use any pair of seats numbered x and y where x *divides* the number y. For example, this cannot be done with $n = 2$, because there are four choices to take $n+1 = 3$ of the $2n = 4$ positions and all of them fail: $\{1, 2, 3\}$ and $\{1, 2, 4\}$ have 1 dividing 2, $\{1, 3, 4\}$ has 1 dividing 3, and $\{2, 3, 4\}$ has 2 dividing 4. They set out to prove that for any positive n this task is impossible. Let $P(n)$ be the statement "In any set of $n + 1$ positions among positions $\{1, 2, \ldots, 2n\}$, some position is chosen whose number divides the number of another position chosen."

(c, 5) Prove the base case, state the inductive hypothesis, and state the inductive goal.

(d, 5) Prove the case of the inductive goal where at least one of the positions $2n+1$ or $2n+2$ is left vacant.

(e, 10XC) Finish the proof by completing the inductive step, in the case where positions $2n+1$ and $2n+2$ are both chosen. (**Hint:** Look at positions $n+1$ and $2n+2$. Argue that we can't use both of them, but that if we use one of them, it doesn't matter which.)

Question 3 (35): Let $\Sigma = \{0, 1\}$ and let R be the regular expression $(01 + 10)^*$.

(a, 10) Construct a λ-NFA N such that $L(N) = L(R)$. For full credit, use the construction from the textbook and lecture.

(b, 10) Using the given construction from the textbook and lecture, build an ordinary NFA N' such that $L(N') = L(N)$.

(c, 10) Using the Subset Construction, build a DFA D such that $L(D) = L(N')$.

(d, 5) Using the minimization construction, or otherwise if you can justify your result, find a minimal DFA M such that $L(M) = L(D)$.

Question 4 (30): The following are fifteen true/false questions, with no explanation needed or wanted, no partial credit for wrong answers, and no penalty for guessing. Some of them refer to the scenarios of the other problems, and/or the entities defined on the supplemental sheet.

- (a) Let p be any position for the Toads and Frogs game, with Toad's turn to start. Then if Toad does *not* have a winning strategy from position p, then Frog *does* have a winning strategy from p.
- (b) If L is not a regular language, and L' is some language such that $L' \subseteq L$, then L' must not be a regular language.
- (c) Consider a BFS and a DFS search of the same finite state space, with both searches using a closed list so that the nodes that have been visited are not added to the open list again. Then BFS will find all reachable states, but DFS might not.
- (d) Let L be a regular language, and let L' be a language made by removing no more than 17 strings from L. Then L' is also a regular language.
- (e) If L is any regular language, then there exists some Turing machine that decides L.
- (f) Let $P(w)$ be a predicate ranging over the strings Σ^*, where $\Sigma = \{a, b\}$. If $P(\lambda)$ is true, and $\forall w : P(w) \rightarrow (P(wa) \wedge P(wb))$, then $\forall w : P(w)$ is true.
- (g) The language $\{a^n b^n : n \geq 0\}$ is Turing recognizable but not Turing decidable.
- (h) Let SC be the set of partially completed 3×3 Sudoku positions (encoded as strings over the alphabet $\{1, 2, 3, 4, 5, 6, 7, 8, 9, x\}$ where an x is not yet filled in) that can be legally completed. Then the language SC is not regular.
- (i) The language SC of the preceding problem is Turing decidable.
- (j) The relation S from Question 1 is not a partial order, but if we define a relation S' so that $S'(x, y) \leftrightarrow (S(x, y) \vee (x = y))$, then S' is a partial order.
- (k) If u is a string in a language L, and v is a string such that u and v are L-equivalent as in the Myhill-Nerode Theorem, then v must also be in L.
- (l) The set of Turing machines that accept their own descriptions is not Turing recognizable.
- (m) Any NFA must contain a death state (a non-final state with all arrows to itself).
- (n) Let $\Sigma = \{a, b, \ldots, z\}$ and let U be the set of strings w that do not have the same letter occurring more than once. Then U is a regular language.
- (o) Any odd natural can be factored as product of zero or more odd prime numbers.

Supplemental Sheet for COMPSCI 250 Final Exam, Fall 2021, D.M. Barrington and G. Parvini

Question 1:

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We assume that both the relations L and S are each antireflexive, antisymmetric, and transitive. We also assume that $\forall x : \forall y : L(x, y) \leftrightarrow S(y, x)$, and that no two dogs have exactly the same size (in symbols, $\forall x : \forall y : (x \neq y) \rightarrow (S(x, y)) \vee S(y, x)))$.

We also define two more predicates on D in terms of the first two, so that $LR(x)$ means "dog x is larger than some retriever" and $SR(x)$ means "dog x is smaller than some retriever".

(a, 10): Translate the following statements as indicated.

- Statement I: (into symbols, using only quantifiers, R, and S) It is not the case that Mango is smaller than some retriever.
- Statement II: (into English) $\forall x : SR(x) \oplus LR(x)$.
- Statement III: (into English) $\exists x : L(b, x) \wedge \neg R(x)$.
- Statement IV: (into symbols) Both Blaze and Kiké are each smaller than every retriever.
- Statement V: (into English) $(S(b, c) \leftrightarrow S(b, i)) \wedge (S(b, k) \oplus S(b, m))$

(b, 10): Using any or all of Statements I-V, determine Blaze's order relative to the other four dogs, by finding the truth values of $S(b, c)$, $S(b, k)$, $S(b, i)$, and $S(b, m)$. (**Note:** This is a bit different from the boolean question on prior exams, as there is only one purely boolean statement. You will get half credit for correctly determining (by a truth table or otherwise) which settings of the truth values make Statement V true. You will need at least some of the other four statements to fully answer the question. You could also fully solve it without Statement V at all.)

(c, 10): Assuming Statement II, and the assumptions and definitions above, determine exactly how many retrievers there are in the set D, and prove your answer.