NAME: ____________________________

COMPSCI 250
Introduction to Computation
Second Midterm Fall 2021

D. A. M. Barrington and G.Parvini 9 November 2021

DIRECTIONS:

- Answer the problems on the exam pages.
- There are five problems on pages 2-8, some with multiple parts, for 100 total points plus 5 extra credit. Probable scale is somewhere around A=95, C=65, but will be determined after we grade the exam.
- Page 9 contains useful definitions and is given to you separately – do not put answers on it!
- If you need extra space use the back of a page.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer.

Question 1 (10): Recall that the Fibonacci function $F(n)$ is defined by the rules $F(0) = 0$, $F(1) = 1$, and for all n with $n > 1$, $F(n) = F(n-1) + F(n-2)$. Prove that for any natural n, $\sum_{k=1}^{n} F(2k-1) = F(2n)$.

Question 2 (15): Blaze keeps careful observations of how many rabbits she sees on each of her daily evening walks. On Day 0 she saw none, on Day 1 she saw one, and on Day 2 she saw none. She continued her observations for several days, and discovered a pattern in the later numbers. Knowing that recurrence relations have been used to model rabbit populations, she saw that for all n with $n \geq 3$, the number $R(n)$ that she saw on day n was always $R(n) = 3 - R(n-1) - R(n-2) - R(n-3)$.

- (a, 5) Assuming that this pattern continues, compute the values of $R(3)$, $R(4)$, $R(5)$, $R(6)$, and $R(7)$.

- (b, 10) Assuming that this pattern continues, how many rabbits will Blaze observe on Day 503? Prove your answer.

Question 3 (20+5XC): While visiting her childhood home, Alice takes a break from her COMPSCI 250 homework to sort a box containing some of her n keepsakes. She plans to eventually place each keepsake into a new separate box, but she does this by a series of steps:

- At each step, she takes one non-empty box and divides its items into one non-empty box of p items and one non-empty box of q items.
- During this time, she takes $p \times q$ minutes to reminisce about those keepsakes.
- (For example, dividing a box of seven items into a box of two and a box of five will take $2 \times 5 = 10$ minutes, while dividing into a box of three and a box of four will take $3 \times 4 = 12$ minutes.)
- Later she wonders which choices would have taken the least or most time to get from the single box of n to the final n boxes with one keepsake each.
- She sees that she has only one option for $n \leq 3$, taking 0 time for $n = 1$, 1 minute for $n = 2$, and 3 minutes for $n = 3$.
- For the next two, she sees that however she makes the choices, the total time is 6 minutes for $n = 4$ and 10 minutes for $n = 5$.
- Will this pattern continue? It looks like $T(n) = \frac{n(n-1)}{2)}$.

- (a, 5) Find the possible choices for $n = 6$ and $n = 7$.

- (b, 15) Prove that for any possible choices of dividing any of the boxes, the total time will be $T(n) = \frac{n(n-1)}{2}$. Use strong induction on all positive naturals, with the base case of $T(1) = 0$.

- (c, 5XC) Suppose that we change the scenario so that Alice spends one minute each time she divides one non-empty box into two non-empty boxes. What is the minimum time for her to get from a single box of n items to n boxes with one item each? Does it matter how she chooses to divide boxes? Prove your answers.

Question 4 (35): Let G be the weighted undirected graph pictured here:

```
    a---3---b---3---c
   / \     / \     / \
  2   1   2   3   2   2
 /     \ /     \ /     \
s---4---d---4---e---4---t
 \     / \     / \     /
  3   2   3   5   3   2
   \ /     \ /     \ /
    f---5---g---1---j
```

We want to find the shortest-path distance (using the weights) from s to t. We will also define a heuristic function $h(x)$ for all nodes x in G, such that $h(x)$ will be the smallest number of edges (ignoring the weights) from x to t.

- (a, 5) Compute the value of $h(x)$ for each of the ten nodes x in G, by any method.

- (b, 5) Which search method, depth-first search or breadth-first search, is useful in solving part (a)? Explain your answer.

- (c, 5) Carry out the search, of the type you chose in (b), for this undirected graph. Describe which nodes enter and leave the open list. Indicate the tree edges and non-tree edges in your search.

- (d, 10) Trace a uniform-cost search with start node s and goal node t, using the given edge weights, in order to find the shortest-path distance from s to t. Indicate which nodes are on the priority queue at each stage of the search.

- (e, 10) Conduct a complete A^* search of G with start node s and goal node t, using the given edge weights and the given values for the heuristic function h. Indicate which nodes are on the priority queue at each stage of the search.

Question 5 (20): The following are ten true/false questions, with no explanation needed or wanted, no partial credit for wrong answers, and no penalty for guessing. Some of them refer to the scenarios of the other problems, and/or the entities defined on the supplemental sheet.

- (a) The smallest natural n_0, such that $2^n \geq n^2$ for all $n \geq n_0$, is 0.

- (b) It follows from the Peano Axioms that if two positive naturals p and q have the same predecessor, then $p = q$.

- (c) If we apply a Uniform-Cost Search algorithm on a directed graph where every edge has weight 1, then the resulting search proceeds like a depth-first search.

- (d) If we carry out a depth-first search from a node s, and the graph contains a directed cycle of more than one node that contains s, then the resulting DFS tree must contain at least one back edge.

- (e) If a state space has finitely many nodes, and no state is ever put on the open list more than once, then a generic search from node s, with t as its goal node, will declare victory if and only if there is a path from s to t.

- (f) There exist some binary associative operations that are commutative, and other binary associative operations that are not commutative.

- (g) In an undirected tree, the number of nodes is always one less than the number of edges.

- (h) If w is any binary string, and oc is the one's complement operation, then $oc(0w) = 1oc(w)$.

- (i) In an A^* search, if we use $h(v) = 0$ for every node v in the graph, then the search behaves exactly like a uniform-cost search.

- (j) If V is the set of nodes in an undirected graph, E is the set of edges, and $deg(v)$ is the degree of a node v, then $\sum_{v \in V} deg(v) = 2|E|$.

Supplemental Page for CS250 Midterm #2, 9 November 2021, Barrington/Parvini

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