CMPSCI 250: Introduction to Computation

Lecture #33: NFA's and the Subset Construction
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Nondeterministic Finite Automata

- Kleene's Theorem: What and Why?
- Nondeterministic Finite Automata
- The Language of an NFA
- The Model of λ-NFA's
- The Subset Construction: NFA's to DFA's
- Applying the Construction to No-aba
- The Validity of the Construction

Kleene's Theorem: What and Why?

- We have now defined two classes of formal languages -- **regular** languages that are denoted by regular expressions, and what we will call **recognizable** languages that are decided by a DFA.
- Kleene's Theorem, the subject of the next several lectures, says that these two classes are the same.

Kleene's Theorem

- Mathematically, it's interesting that two classes with such different definitions should turn out to coincide -- it suggests that the class is important.
- But the theorem also has practical consequences.
- A class of languages is closed under an operation if applying the operation to elements of the class results in another element.

Kleene's Theorem

- It's easy to see that the regular languages are closed under union, concatenation, and star, and that the recognizable languages are closed under complement and intersection.
- The theorem tells us that *both* classes have *all* these closure properties.
- The efficient way to test whether a string is in a regular language is to create the DFA for the language and run it on the string.

Nondeterminism

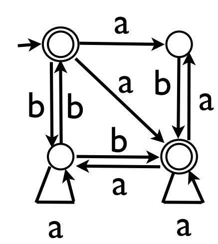
- DFA's are **deterministic** in that the same input always leads to the same output.
- Some algorithms are not deterministic because they are randomized, but here we will consider "algorithms" that are not deterministic because they are **underdefined** -- given a single input, more than one output is possible.
- We had an example of such an algorithm with our generic search, which didn't say *which* element came off the open list when we needed a new one.

Nondeterministic Finite Automata

- Formally, a **nondeterministic finite automaton** or **NFA** has an alphabet, state set, start state, and final state just like a DFA.
- But instead of the transition function δ, it has a **transition relation** $\Delta \subseteq Q \times \Sigma \times Q$. If $(p, a, q) \in \Delta$, the NFA *may* move to state q if it sees the letter a while in state p.

Drawing an NFA

- We draw an NFA like a DFA, with an a-arrow from p to q whenever (p, a, q) $\in \Delta$.
- The NFA no longer has the rule that there must be exactly one arrow for each letter out of each state -- there may be more than one, or none.

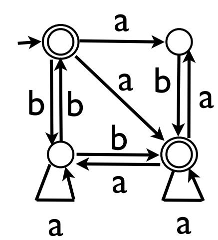


The Language of an NFA

- We can no longer say what the NFA *will* do when reading a string, only what it *might* do. The language of an NFA N is defined to be the set {w: w might be accepted by N}.
- More formally, we define a relation $\Delta^* \subseteq Q \times \Sigma^* \times Q$ so that the triple (p, w, q) is in Δ^* if and only if N *might* go from p to q while reading w.
- Then $w \in L(N) \leftrightarrow (i, w, f) \in \Delta^*$ for some final state $f \in F$.

Clicker Question #1

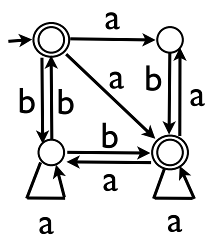
- A string w is in the language of this NFA if it is possible to follow a path with the letters of w from the start state to a final state. Which string is not in L(N)?



- (a) baa
- (b) aab
- (c) aaa
- (d) bbb

Answer #1

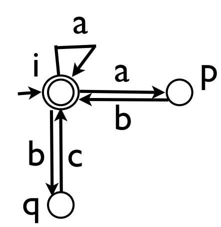
- A string w is in the language of this NFA if it is possible to follow a path with the letters of w from the start state to a final state. Which string is not in L(N)?



- *(a) baa*
- (b) aab
- (c) aaa
- (d) bbb *(whoops, bbb $\notin$ L(N) also)*

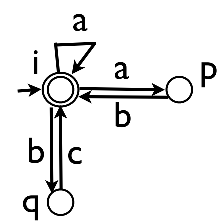
An NFA Example

- Consider the NFA N with state set {i, p, q}, start state i, final state set {i}, alphabet {a, b, c}, and Δ = {(i, a, i), (i, a, p), (p, b, i), (i, b, q), (q, c, i)}.
- This is nondeterministic because there are two a-moves out of i, and several situations with no move at all.



An NFA Example

- Here L(N) is the regular language $(a + ab + bc)^*$, because any path from i to itself must consist of pieces labeled a, ab, or bc.



- It is not immediately clear how, for a larger NFA, we could determine whether a particular string was in L(N). Our method will be to turn N into a DFA.

Interpretations of Nondeterminism

- Because we can't speak clearly of "what happens when we run N on w", we need other ways to think of the action of an NFA.
- In our proofs, we will just replace "$w \in L(N)$" by "$\exists f$: $(i, w, f) \in \Delta^*$" and argue about the possible w-paths in the graph of N.

Interpretations of Nondeterminism

- Suppose the NFA makes a choice uniformly at random whenever it has more than one option. This makes it a **Markov process** in the language of CMPSCI 240.
- In this case $w \in L(N)$ if and only if the probability that N goes to a final state on w is positive. If there is a path, there is a nonzero probability of N taking it, and if there is no path, of course it cannot possibly reach a final state.

Interpretations of Nondeterminism

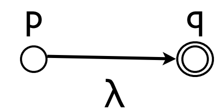
- Another interpretation has us **fork a process** whenever N is faced with a choice. One process takes each choice, and if *any* of the processes reaches a final state when it is done reading w, then w $\in$ L(N).
- *"When you come to a fork in the road... take it." (Y. Berra)*

The Model of λ-NFA's

- The main reason to use NFA's is that they are easier to design in many situations when we have some other definition of the language.
- Often we will find it convenient to give the NFA the option to jump from one state to another *without reading a letter*.
- A **λ-move** is a transition (p, λ, q) that allows a **λ-NFA** to do just that.

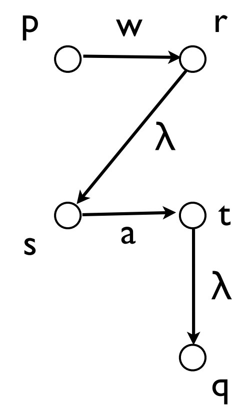
The Model of λ-NFA's

- We need to redefine the type of Δ, so that it is a subset of $Q \times (\Sigma \cup \{\lambda\}) \times Q$.
- In the diagram, this transition is represented by an arrow from p to q labeled with λ.



Paths in a λ-NFA

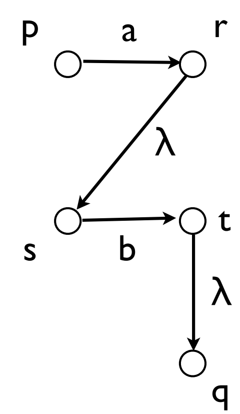
- Formally Δ^* is now more complicated to define. We say that $(p, \lambda, q) \in \Delta^*$ if there is a path of λ-moves from p to q.
- Then we define $\Delta^*(p, wa, q)$ to be true if and only if there exist states r, s, and t such that (p, w, r), (r, λ, s) and (t, λ, q) are all in Δ^*, and (s, a, t) is in Δ.



Paths in a λ-NFA

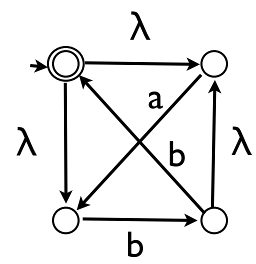
- What this means is that $\Delta^*(p, w, q)$ is true if and only if there exists a path from p to q such that the letters on the path, read in order, spell out w.
- There may be any number of λ-moves in the path as well.
- (Thus we don't even know how long the path from p to q might be.)

$(p, ab, q) \in \Delta^*$



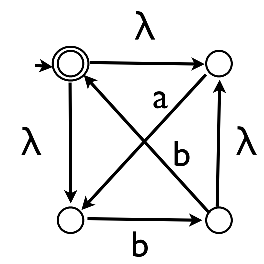
Clicker Question #2

- Which of these strings is not in the language of this λ-NFA?
- (a) λ
- (b) bab
- (c) abbbb
- (d) Trick question: all three are in the language.



Answer #2

- Which of these strings is not in the language of this λ-NFA?
- (a) λ
- *(b) bab (can only reach final state with two consecutive b's)*
- (c) abbbb
- (d) Trick question: all three are in the language.

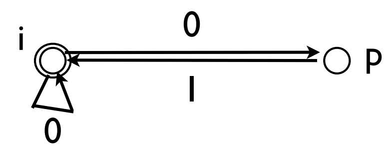


The Subset Construction

- Next lecture we'll see how to convert λ-NFA's to ordinary NFA's.
- Now, though, we will convert ordinary NFA's to DFA's using the **Subset Construction**.
- Given an NFA N with state set Q, we will build a DFA D whose states will be *sets of states* of N -- formally, D's state set is the **power set** of Q.

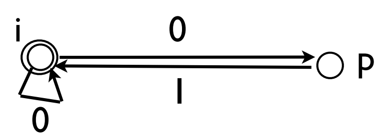
The Subset Construction

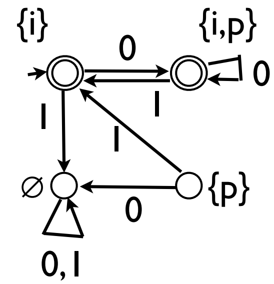
- Here's an example of an NFA N for the language $(0 + 01)^*$, with two states i and p, start state i, final state set {i}, and transitions (i, 0, i), (i, 0, p), and (p, 1, i).
- At the start of its run, N must be in state i. If the first letter is 0, then it might be in either state i or p after reading the 0. If the first letter is 1, there is no run of N that reads that letter.



The Subset Construction

- Our DFA D has states ∅, {i}, {p}, and {i, p}.
- Its start state is {i}, its final states are {i} and {i, p}, and we have δ({i}, 0) = {i, p}, δ({i}, 1) = ∅, δ({i, p}, 0) = {i, p}, δ({i, p}, 1) = {i}, δ({p}, 0) = ∅, δ({p}, 1) = {i}, and δ(∅, a) = ∅ for both letters.





Details of the Construction

- The general construction works just like this example.
- The start state of D is {i}, where i is the start state of N.
- The final state set of D is the set of all states of D that contain final states of N, since we want D to accept if and only if N *can* accept.

Details of the Construction

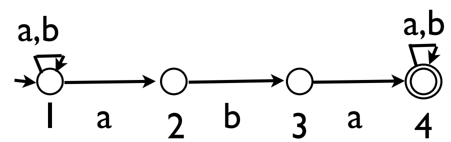
- In general, we need to define $\delta(S, a)$, where S is a state of D, meaning that S is a set of states of N.
- S represents the possible places N might be before reading the a. The set $T = \delta(S, a)$ will be the set of all states q such that the transition (s, a, q) is in Δ for some $s \in S$.
- In the graph, we take the set of destinations of all the a-arrows that start from a state of S.

Details of the Construction

- The most common mistake in computing δ comes when one of the states in S has no a-arrows out of it.
- Students often think that $\varnothing$ must now be part of $\delta(S, a)$. But in fact $\delta(S, a)$ is the *union* of the sets $\{q: \Delta(s, a, q)\}$ for each $s \in S$.
- So the empty set is part of the result, but doesn't show up in the description of the result because unioning with $\varnothing$ is the identity operation on sets.

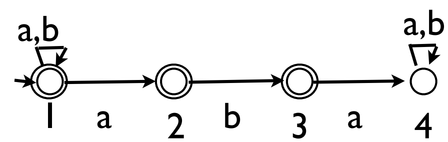
Applying This to No-aba

- The language Yes-aba has an easy regular expression $\Sigma^*aba\Sigma^*$. From this expression we can build an NFA N with state set {1, 2, 3, 4}, start state 1, final state set {4}, and Δ = {(1, a, 1), (1, b, 1), (1, a, 2), (2, b, 3), (3, a, 4), (4, a, 4), (4, b, 4)}.
- But what if we want a machine for No-aba? Switching the final and non-final states of N will not do -- can you see why?



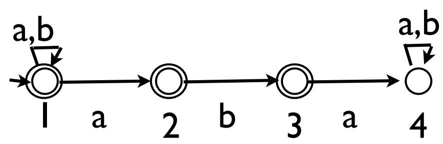
Clicker Question #3

- What is the language of this NFA?
- (a) $\emptyset^* + a + ab$
- (b) $\emptyset^* + (a + b)^*a + (a + b)^*ab$
- (c) $(a + b)^*$
- (d) All three expressions are correct.



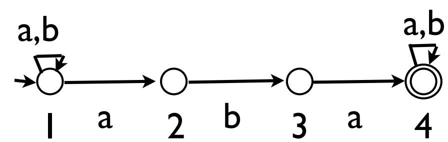
Answer #3

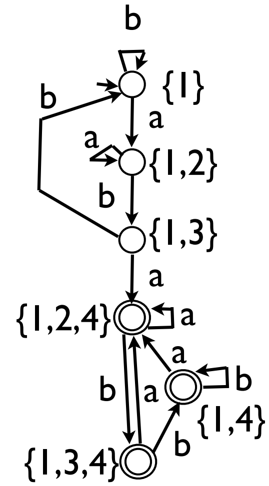
- What is the language of this NFA?
- (a) $\varnothing^* + a + ab$
- (b) $\varnothing^* + (a + b)^*a + (a + b)^*ab$
- *(c) $(a + b)^*$ (any string has a path from 1 to 1)*
- (d) All three expressions are correct.



Applying This to No-aba

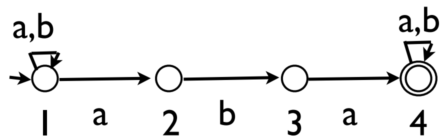
- The best way to get a DFA for No-aba is to first get one for Yes-aba.
- We begin with the start state {1} and compute δ({1}, a) = {1, 2} and δ({1}, b) = {1}. Then we compute δ({1, 2}, a) = {1, 2} and δ({1, 2}, b) = {1, 3}.

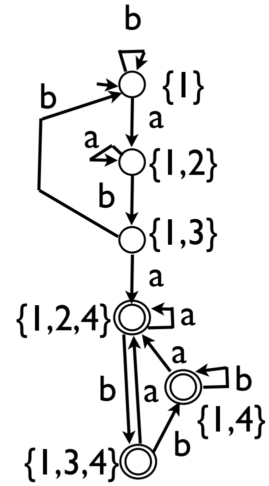




Applying This to No-aba

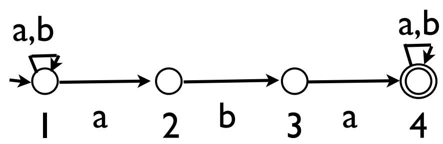
- Since {1, 3} is new, we must compute δ({1, 3}, a) = {1, 2, 4} and δ({1, 3}, b) = {1}.
- Then we get δ({1, 2, 4}, a) = {1, 2, 4} and δ({1, 2, 4}, b) = {1, 3, 4}. Not done yet!
- We have δ({1, 3, 4}, a) = {1, 2, 4} and δ({1, 3, 4}, b) = {1, 4}.

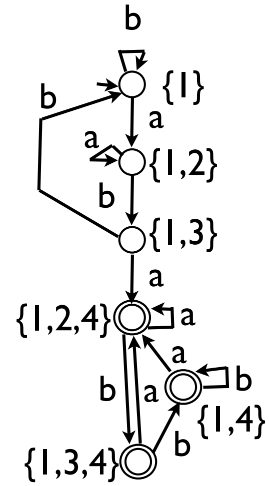




Applying This to No-aba

- Finally, with $\delta(\{1, 4\}, a) = \{1, 2, 4\}$ and $\delta(\{1, 4\}, b) = \{1, 4\}$, we're done -- we have all reachable states.
- If we minimized this DFA, the three final states would merge into one. This gives us our four-state DFA for Yes-aba, from which we can get one for No-aba.





Validity of the Construction

- How can we prove that for any NFA N, the DFA D that we construct in this way has L(D) = L(N)?
- The key property of D is that for any string w, $\delta^*(\{i\}, w)$ is exactly the set of states $\{q: \Delta^*(i, w, q)\}$ that could be reached from i on a w-path.
- We prove this property by induction -- it is clearly true for λ (though if we had λ-moves it would not be).

Validity of the Construction

- If we assume that $\delta^*(\{i\}, w) = \{q: \Delta^*(i, w, q)\}$, we can then prove $\delta^*(\{i\}, wa) = \{r: \Delta^*(i, wa, r)\}$ for an arbitrary letter a, using the inductive definition of δ^* in terms of δ, of δ in terms of Δ, and of Δ^* in terms of Δ.
- Once this is done, it is clear that $w \in L(D) \leftrightarrow$

  $\exists f: f \in \delta^*(\{i\}, w) \leftrightarrow \exists f: \Delta^*(i, w, f) \leftrightarrow w \in L(N)$.
- Note that in general D could have 2^k states when N has k states. But if we leave out unreachable states, D could be much smaller.