1. Consider the following algorithm for the Steiner Tree Problem

**Algorithm**

- Let \( \{s_1, s_2, ..., s_{|S|}\} \) be any ordering of the terminals.
- Let \( T \leftarrow \{s_1\} \)
- For \( i = 2 \) to \(|S|\)
  - Let \( P_i \) be the shortest path connecting \( s_i \) to \( T \).
  - Add \( P_i \) to \( T \)

Show that the above algorithm gives a \( \lceil \log_2 |S| \rceil \) approximation factor for the Steiner Tree Problem

*Points 30*

2. Given a complete undirected graph \( G = (V, E) \) with nonnegative edge weights, where edge weights satisfy triangle inequality, and \( k \) colors \( c_1, c_2, ..., c_k \), find an assignment \( \phi \) of colors to vertices such that

- Each vertex is assigned exactly one color.
- Let \( d_r(v) \) be the distance to nearest node from \( v \) that is assigned color \( c_r \). Let \( D_v = \max_{r=1}^{k} d_r(v) \). The assignment must minimize the maximum \( D_v \) over all \( v \), that is find \( \phi \) such that \( \max_v D_v \) is minimized.

(i) Show the above problem is NP-Hard.
(ii) Obtain a 3-approximation algorithm for the above problem.

*Points 10+30=40*

3. Given an undirected graph \( G = (V, E) \), find a spanning tree \( T \) of \( G \) that has maximum number of leaves.

(i) Show the above problem is NP-Hard.
(ii) Consider the following local search heuristic.

- Start with any arbitrary spanning tree \( T \)
- While there are edges \( e \in T \) and \( f \notin T \) such that removing \( e \) from \( T \) and including \( f \) creates a spanning tree with more leaves, swap\((e, f)\)
Return T when no such improving swaps exist.

Show that the above local search algorithm gives an approximation factor of at most 10.

To obtain the above result, first prove the following claims. Let $n_i$ denote the number of nodes of degree $i$ in $T$ and let $n_{\geq i}$ denote the number of nodes of degree at least $i$ in $T$.

(a) Prove for any tree $T$, $n_{\geq 3}(T) < n_1(T)$.
(b) Define a 2-path to be a maximal (longest) path such that all internal nodes in the path have degree exactly 2 in $T$. Let $n_{2Paths}$ denote the number of such maximal 2-paths. Show $n_{2Paths} < 2n_1(T)$.

Use (a) and (b) to establish an approximation factor of at most 10 for the local search algorithm.

Points 10+40

4. Obtain an FPTAS for the following problem.

Given $n$ positive integers $a_1 < a_2 < \ldots < a_n$, find two disjoint nonempty subsets $S_1, S_2 \subseteq \{1, 2, \ldots, n\}$ with $\sum_{i \in S_1} a_i \geq \sum_{i \in S_2} a_i$, such that the ratio

$$\frac{\sum_{i \in S_1} a_i}{\sum_{i \in S_2} a_i}$$

is minimized.

Points 30