Bounded Degree Spanning Tree using Iterative Relaxation

Barna Saha

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Bounded Degree Spanning Tree

Iterative Minimum Spanning Tree Algorithm

Iterative Algorithm for Degree Bounded Spanning Tree

Primal Dual Methods for Algorithm Design
Problem Definition

Problem
We are given an undirected graph $G = (V, E)$ with a degree upper bound $B_v$ on each vertex $v$, and cost on edges $c : E \rightarrow \mathbb{R}^+$. The task is to find a minimum cost spanning tree which satisfies all the degree bounds.

- The problem is NP-hard. Since, if every node has degree upper bound 2, and we can solve it exactly then we know whether the graph $G$ has a hamiltonian path or not.
- We will show an algorithm which returns a spanning tree $T$ of cost at most $OPT$ where $d_T(v) \leq B_v + 1$ for all $v$, where $d_T(v)$ denotes the degree of $v$ in $T$. 
Spanning Tree Polytope with Degree Bounds

- Given a subset \( S \subseteq V \), we use \( E(S) \) to denote all edges with two endpoints in \( S \).
- We use \( \delta(S) \) to denote all edges with exactly one endpoint in \( S \).

Here is the natural LP relaxation.

\[
\text{minimize } c(x) = \sum_{e \in E} c_e x_e \\
\text{subject to } x(E(V)) = |V| - 1 \\
x(E(S)) \leq |S| - 1 \quad \forall S \subset V \\
x(\delta(v)) \leq B_v \quad \forall v \in V \\
x_e \geq 0 \quad \forall e \in E
\]

We can solve this LP using ellipsoid method, as there exists a polynomial time separation oracle (Read chapter 11.2 of Shmoys and Williamson)
Spanning Tree Polytope

\[
\text{minimize } \quad c(x) = \sum_{e \in E} c_e x_e \\
\text{subject to } \quad x(E(V)) = |V| - 1 \\
\quad x(E(S)) \leq |S| - 1 \quad \forall S \subset V \\
\quad x_e \geq 0 \quad \forall e \in E
\]

(LP-MST)

We will first show an iterative procedure to obtain a minimum spanning tree of \( G \).
Iterative Minimum Spanning Tree Algorithm

- Initialization $F \leftarrow \phi$
- While $V(G) \neq \phi$ do
  - Find a basic optimal solution $x^*$ of LP-MST($G$) and remove every edge $e$ with $x^*_e = 0$ from $G$.
  - Find a vertex $v$ with at most one edge $e = uv$ incident at it, and update $F \rightarrow F \cup \{e\}$, $G \rightarrow G \setminus \{v\}$.
- Return $F$
Proof: Step 1

Lemma

If the Iterative MST Algorithm terminates then the algorithm returns an optimal solution.

Proof.

If the algorithm finds a degree 1 vertex $v$, then $x(\delta(v)) \geq 1$. Hence if edge $e = (u, v)$ incident on $v$, $x^*_e = 1$.

$$x(E(S)) = |V| - 1, x(E(S \setminus v)) \leq |V| - 1 - 1 = |V| - 2.$$  

$$x(\delta(v)) = x(E(S)) - x(E(S \setminus v)) \geq 1$$

- Given any spanning tree $T'$ of $G' = G \setminus \{v\}$, $T' \cup \{e\}$ is a spanning tree of $G$.
- Solve the problem recursively on $G'$. 
Proof: Step 1

Lemma
If the Iterative MST Algorithm terminates then the algorithm returns an optimal solution.

▶ Solve the problem recursively on $G'$.
  ▶ Restriction of $x^*$ to $G'$ (denoted $x^*_\text{res}$) is a feasible solution for LP-MST($G'$).
  ▶ Inductively, we have an optimal solution $F'$ for $G'$.
  ▶ Hence, if $F$ is the final solution, $F = F' \cup e$.

$$c(F) = c(F') + c_e \leq c(x^*_\text{res}) + c_e = c(x^*)$$
Proof: Step 2

Lemma

For any basic solution $x^*$ of $LP-MST(G)$ with support $E^* = \{ e \mid x_e^* > 0 \}$, there exists a vertex $v$ with $\text{deg}_{E^*}(v) = 1$

Proof Sketch

- Show there are at most $|V| - 1$ linearly independent tight constraints.
- By definition of basic solution, this implies there are at most $|V| - 1$ edges with fractional values.
- This implies there exists one vertex with degree 1.
Proof: Step 2

Lemma

For any basic solution $x^*$ of LP-MST($G$) with support $E^* = \{ e \mid x^*_e > 0 \}$, there exists a vertex $v$ with $\deg_{E^*}(v) = 1$

Proof Sketch

▶ Show there are at most $|V| - 1$ linearly independent tight constraints.
▶ By definition of basic solution, this implies there are at most $|V| - 1$ edges with fractional values.
▶ This implies there exists one vertex with degree 1.
Proof: Step 2: Notations

- Let $\mathcal{F} = \{ S \mid x^*(E(S)) = |S| - 1 \}$ be the set of tight constraints.

- For a set $S \subseteq V$, the corresponding constraint $x^*(E(S)) \leq |S| - 1$ defines a vector in $\mathbb{R}^{|E|}$: the vector has a 1 corresponding to each edge $e \in E(S)$, and 0 otherwise.
  - characteristic vector of $E(S)$: denoted $\chi_{E(S)}$
Proof: Step 2: Notations

- Let \( \mathcal{F} = \{ S \mid x^*(E(S)) = |S| - 1 \} \) be the set of tight constraints.

- For a set \( S \subseteq V \), the corresponding constraint \( x^*(E(S)) \leq |S| - 1 \) defines a vector in \( \mathbb{R}^{|E|} \): the vector has a 1 corresponding to each edge \( e \in E(S) \), and 0 otherwise.
  - characteristic vector of \( E(S) \): denoted \( \chi_{E(S)} \)

- \( \text{span}(\mathcal{F}) \): vector space generated by linear combination of vectors \( \{ \chi_{E(S)} \mid S \in \mathcal{F} \} \)
Proof: Step 2: Notations

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- For a set $S \subseteq V$, the corresponding constraint $x^*(E(S)) \leq |S| - 1$ defines a vector in $\mathbb{R}^{|E|}$: the vector has a 1 corresponding to each edge $e \in E(S)$, and 0 otherwise.
  - characteristic vector of $E(S)$: denoted $\mathcal{X}_{E(S)}$

- $\text{span}(\mathcal{F})$: vector space generated by linear combination of vectors $\{ \mathcal{X}_{E(S)} \mid S \in \mathcal{F} \}$

- Two sets $X$ and $Y$ are intersecting, if $X \cap Y$, $X - Y$, $Y - X$ are nonempty. A family of sets is laminar if no two sets are intersecting.
Proof: Step 2: Notations

- Let $\mathcal{F} = \{ S \mid x^*(E(S)) = |S| - 1 \}$ be the set of tight constraints.
- For a set $S \subseteq V$, the corresponding constraint $x^*(E(S)) \leq |S| - 1$ defines a vector in $\mathbb{R}^{|E|}$: the vector has a 1 corresponding to each edge $e \in E(S)$, and 0 otherwise.
  - characteristic vector of $E(S)$: denoted $\chi_{E(S)}$
- $\text{span}(\mathcal{F})$: vector space generated by linear combination of vectors $\{\chi_{E(S)} \mid S \in \mathcal{F}\}$
- Two sets $X$ and $Y$ are intersecting, if $X \cap Y$, $X - Y$, $Y - X$ are nonempty. A family of sets is laminar if no two sets are intersecting.
- We can obtain a laminar family $\mathcal{L} \subseteq \mathcal{F}$ such that $\text{span}(\mathcal{L}) = \text{span}(\mathcal{F})$. 
Proof: Step 2

- Show there are at most $|V| - 1$ linearly independent tight constraints.
  - Show $|L| \leq |V| - 1$ where each $S \in L$ has cardinality at least 2.
    - Show by induction. There are $|V|$ ground elements. The size of a laminar family each having cardinality at least 2 is bounded by $|V| - 1$. 
Proof: Step 2

Show a laminar family $\mathcal{L}$ exists with $\text{span}(\mathcal{L}) = \text{span}(\mathcal{F})$. 
Lemma

If $S, T \in \mathcal{F}$ and $S \cap T \neq \emptyset$, then both $S \cap T$ and $S \cup T$ are in $\mathcal{F}$. Furthermore, $\chi_E(S) + \chi_E(T) = \chi_E(S \cap T) + \chi_E(S \cup T)$.

- Implies the 4 constraints corresponding to $S, T, S \cup T, S \cap T$ are not all tight and linearly independent.
- We do not need to consider (say) $T$.
- We therefore have three laminar constraints.
Proof: Step 2

Lemma
If $S, T \in \mathcal{F}$ and $S \cap T \neq \emptyset$, then both $S \cap T$ and $S \cup T$ are in $\mathcal{F}$. Furthermore, $\chi_E(S) + \chi_E(T) = \chi_E(S \cap T) + \chi_E(S \cup T)$.

Proof.
As $S \cap T \neq \emptyset$, we have:

$$|S| - 1 + |T| - 1$$

$$= |S \cap T| - 1 + |S \cup T| - 1 \quad (|S| + |T| - |S \cap T| = |S \cup T|)$$

$$\geq x^*(E(S \cap T)) + x^*(E(S \cup T)) \quad (x^*(E(S)) \leq |S| - 1 \text{ for all } S \subset V)$$

$$\geq x^*(E(S)) + x^*(E(T)) \quad \text{(not counting edges that go from } S \text{ to } T)$$

$$= |S| - 1 + |T| - 1 \quad \text{used } x^*(E(S)) = |S| - 1, x^*(E(T)) = |T| - 1$$

All inequalities must be equalities. \qed
Proof: Step 2

Lemma

If \( S, T \in \mathcal{F} \) and \( S \cap T \neq \emptyset \), then both \( S \cap T \) and \( S \cup T \) are in \( \mathcal{F} \). Furthermore, \( x_{E(S)} + x_{E(T)} = x_{E(S \cap T)} + x_{E(S \cup T)} \).

Proof.

As \( S \cap T \neq \emptyset \), we have:

\[
|S| - 1 + |T| - 1
= |S \cap T| - 1 + |S \cup T| - 1 \quad (|S| + |T| - |S \cap T| = |S \cup T|)
\geq x^*(E(S \cap T)) + x^*(E(S \cup T)) \quad (x^*(E(S)) \leq |S| - 1 \text{ for all } S \subseteq V)
\geq x^*(E(S)) + x^*(E(T)) \quad \text{(not counting edges that go from } S \text{ to } T)
= |S| - 1 + |T| - 1 \quad \text{used } x^*(E(S)) = |S| - 1, x^*(E(T)) = |T| - 1
\]

Constraints for \( S \cup T \) and \( S \cap T \) are also tight.
\( x^*(E(S \cup T)) = |S \cup T| - 1 \), and \( x^*(E(S \cap T)) = |S \cap T| - 1 \)
Proof: Step 2

Lemma
If \( S, T \in \mathcal{F} \) and \( S \cap T \neq \emptyset \), then both \( S \cap T \) and \( S \cup T \) are in \( \mathcal{F} \). Furthermore, \( \chi_{E(S)} + \chi_{E(T)} = \chi_{E(S \cap T)} + \chi_{E(S \cup T)} \).

Proof.
As \( S \cap T \neq \emptyset \), we have:

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|S| - 1 + |T| - 1 \\
= |S \cap T| - 1 + |S \cup T| - 1 \quad (|S| + |T| - |S \cap T| = |S \cup T|) \\
\geq x^*(E(S \cap T)) + x^*(E(S \cup T)) \quad (x^*(E(S)) \leq |S| - 1 \text{ for all } S \subset V) \\
\geq x^*(E(S)) + x^*(E(T)) \quad \text{(not counting edges that go from } S \text{ to } T) \\
= |S| - 1 + |T| - 1 \quad \text{used } x^*(E(S)) = |S| - 1, x^*(E(T)) = |T| - 1
\]

No edge from \( S \) to \( T \). Hence
\( \chi_{E(S)} + \chi_{E(T)} = \chi_{E(S \cap T)} + \chi_{E(S \cup T)} \).
\qed
Lemma

If $\mathcal{L}$ is a maximal laminar subfamily of $\mathcal{F}$, then $\text{span}(\mathcal{L}) = \text{span}(\mathcal{F})$

Proof.

- Suppose $S \in \mathcal{F}$ but $\chi_{E(S)} \not\in \text{span}(\mathcal{L})$. Choose one such $S$ that intersects as few sets of $\mathcal{L}$ as possible.
- Let $S \cap T \neq \emptyset$ and $T \in \mathcal{L}$
- We have $S \cap T$ and $S \cup T$ are in $\mathcal{F}$, but both of them are not in $\mathcal{L}$, otherwise $S$ will be in $\text{span}(\mathcal{F})$.
- Both $S \cup T$ and $S \cap T$ intersects fewer sets in $\mathcal{L}$ than $S$–contradiction.
Iterative Algorithm for Degree Bounded Spanning Tree

In the following we assume degree bounds are only given for vertices in \( W \).

\[
\text{minimize } \quad c(x) = \sum_{e \in E} c_e x_e \\
\text{subject to } \quad x(E(V)) = |V| - 1 \quad \text{(Type-1)}
\]

\[
x(E(S)) \leq |S| - 1 \quad \forall S \subset V \quad \text{(Type-2)}
\]

\[
x(\delta(v)) \leq B_v \quad \forall v \in W \quad \text{(Type-Degree)}
\]

\[
x_e \geq 0 \quad \forall e \in E \quad \text{(9)}
\]

Initially \( W = V \)
Iterative Algorithm for Degree Bounded Spanning Tree

In the following we assume degree bounds are only given for vertices in $W$.

- **Initialization** $F \leftarrow \emptyset$
- While $V(G) \neq \emptyset$ do
  - Find a basic optimal solution $x^*$ of LP-MBDST($G, B, W$), and remove every edge $e$ with $x_e^* = 0$ from $G$. Let the support of $x^*$ be $E^*$.
  - If there exists a vertex $v \in V$, such that there us at most one edge $e = (u, v)$ incident at $v$ in $E^*$, then update $F \leftarrow F \cup \{e\}$, $G \leftarrow G \setminus \{v\}$, $W \leftarrow W \setminus \{v\}$, and also update $B$ by setting $B_u \setminus B_u - 1$.
  - If there exists a vertex $v \in W$ such that $\deg_{E^*}(v) \leq 3$ then update $W \leftarrow W \setminus \{v\}$.
- Return $F$
Iterative Algorithm for Degree Bounded Spanning Tree

Proof Ideas

- Show the number of linearly independent tight constraint is at most $|\mathcal{L}| + |\mathcal{W}|$ where $\mathcal{L}$ is a maximal laminar family of tight constraints of Type-1.

- Argue $|\mathcal{L}| + |\mathcal{W}| \leq |V| + |\mathcal{W}| - 1$.

- If all vertices have degree at least 4 in $\mathcal{W}$ and 2 in $V - \mathcal{W}$ then total degree $\geq 2(|V - \mathcal{W}|) + 4|\mathcal{W}| = 2(|V| + |\mathcal{W}|)$.

- Hence, number of nonzero variables is at least $|V| + |\mathcal{W}|$.

- CONTRADICTION.

- Hence, we either have a leaf vertex, or a Type-Degree constraint to drop.
Iterative Algorithm for Degree Bounded Spanning Tree

Proof Ideas

- Show the number of linearly independent tight constraint is at most $|\mathcal{L}| + |W|$ where $\mathcal{L}$ is a maximal laminar family of tight constraints of Type-1.
- Argue $|\mathcal{L}| + |W| \leq |V| + |W| - 1$.

CONTRACTION.

Hence, we either have a leaf vertex, or a Type-Degree constraint to drop.
Proof Ideas

▶ Show the number of linearly independent tight constraint is at most $|\mathcal{L}| + |W|$ where $\mathcal{L}$ is a maximal laminar family of tight constraints of Type-1.

▶ Argue $|\mathcal{L}| + |W| \leq |V| + |W| - 1$.

▶ If all vertices have degree at least 4 in $W$ and 2 in $V - W$ then total degree $\geq 2(|V - W|) + 4|W| = 2(|V| + |W|)$. Hence, number of nonzero variables is at least $|V| + |W|$.

CONTRADICTION.

▶ Hence, we either have a leaf vertex, or a Type-Degree constraint to drop.
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Proof Ideas

▷ Show the number of linearly independent tight constraint is at most $|\mathcal{L}| + |W|$ where $\mathcal{L}$ is a maximal laminar family of tight constraints of Type-1.

▷ Argue $|\mathcal{L}| + |W| \leq |V| + |W| - 1$.

▷ If all vertices have degree at least 4 in $W$ and 2 in $V - W$ then total degree $\geq 2(|V - W|) + 4|W| = 2(|V| + |W|)$. Hence, number of nonzero variables is at least $|V| + |W|$.

▷ CONTRADICTION.
Iterative Algorithm for Degree Bounded Spanning Tree

Proof Ideas

▷ Show the number of linearly independent tight constraint is at most $|\mathcal{L}| + |\mathcal{W}|$ where $\mathcal{L}$ is a maximal laminar family of tight constraints of Type-1.

▷ Argue $|\mathcal{L}| + |\mathcal{W}| \leq |V| + |W| - 1$.

▷ If all vertices have degree at least 4 in $W$ and 2 in $V - W$ then total degree $\geq 2(|V - W|) + 4|W| = 2(|V| + |W|)$. Hence, number of nonzero variables is at least $|V| + |W|$.

▷ CONTRADICTION.

▷ Hence, we either have a leaf vertex, or a Type-Degree constraint to drop.
Recap

- Deterministic LP Rounding (Vertex Cover, Set Cover)
- Independent Randomized Rounding (Set Cover)
- Dependent Rounding using Basic Solution (Unrelated Parallel Machine Scheduling)
- Iterative Relaxation (Matching, Spanning Tree, Unrelated Parallel Machine Scheduling, Degree Bounded Spanning Tree)
Techniques

- Deterministic LP Rounding (Vertex Cover, Set Cover)
- Independent Randomized Rounding (Set Cover)
- Dependent Rounding using Basic Solution (Unrelated Parallel Machine Scheduling)
- Iterative Relaxation (Matching, Spanning Tree, Unrelated Parallel Machine Scheduling, Degree Bounded Spanning Tree)
- Primal Dual Method
Primal Dual Methods

Primal Dual Methods for Approximation Algorithm
We want to achieve a lower bound on the LP objective value.

minimize $6x_1 + 4x_2 + 2x_3$
subject to $4x_1 + 2x_2 + x_3 \geq 5,$
$x_1 + x_2 \geq 3,$
$x_2 + x_3 \geq 4,$
$x_1, x_2, x_3 \geq 0$

Define a variable for each constraint.
LP-Duality

We want to achieve a lower bound on the LP objective value.

minimize $6x_1 + 4x_2 + 2x_3$
subject to $4x_1 + 2x_2 + x_3 \geq 5, y_1$
$x_1 + x_2 \geq 3, y_2$
$x_2 + x_3 \geq 4, y_3$

$x_1, x_2, x_3 \geq 0$

Set $y_1, y_2, y_3 \geq 0$. Then it must hold that
$y_1(4x_1 + 2x_2 + x_3) + y_2(x_1 + x_2) + y_3(x_2 + x_3) \geq 5y_1 + 3y_2 + 4y_3$
LP-Duality

We want to achieve a lower bound on the LP objective value.

\[
\begin{align*}
\text{minimize} & \quad 6x_1 + 4x_2 + 2x_3 \\
\text{subject to} & \quad 4x_1 + 2x_2 + x_3 \geq 5, y_1 \\
& \quad x_1 + x_2 \geq 3, y_2 \\
& \quad x_2 + x_3 \geq 4, y_3 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

- Set \( y_1, y_2, y_3 \geq 0 \). Then it must hold that
  \[
y_1(4x_1 + 2x_2 + x_3) + y_2(x_1 + x_2) + y_3(x_2 + x_3) \geq 5y_1 + 3y_2 + 4y_3
\]
- \( 5y_1 + 3y_2 + 4y_3 \) will serve as a lower bound for LP objective value if
  \[
  6x_1 + 4x_2 + 2x_3 \geq y_1(4x_1 + 2x_2 + x_3) + y_2(x_1 + x_2) + y_3(x_2 + x_3)
\]
- Rearranging
  \[
  6x_1 + 4x_2 + 2x_3 \geq x_1(4y_1 + y_2) + x_2(2y_1 + y_2 + y_3) + x_3(y_1 + y_3)
\]
LP-Duality

We want to achieve a lower bound on the LP objective value.

minimize $6x_1 + 4x_2 + 2x_3$
subject to $4x_1 + 2x_2 + x_3 \geq 5, y_1$
$x_1 + x_2 \geq 3, y_2$
$x_2 + x_3 \geq 4, y_3$
$x_1, x_2, x_3 \geq 0$

$\triangleright$ $5y_1 + 3y_2 + 4y_3$ will serve as a lower bound for LP objective value if

$6x_1 + 4x_2 + 2x_3 \geq x_1(4y_1 + y_2) + x_2(2y_1 + y_2 + y_3) + x_3(y_1 + y_3)$

$y_1, y_2, y_3 \geq 0$

$\triangleright$ $4y_1 + y_2 \leq 6, 2y_1 + y_2 + y_3 \leq 4, y_1 + y_3 \leq 2$
**LP-Duality**

We want to achieve a lower bound on the LP objective value.

\[
\text{minimize} \quad 6x_1 + 4x_2 + 2x_3 \\
\text{subject to} \quad 4x_1 + 2x_2 + x_3 \geq 5, \ y_1 \\
\quad x_1 + x_2 \geq 3, \ y_2 \\
\quad x_2 + x_3 \geq 4, \ y_3 \\
\quad x_1, x_2, x_3 \geq 0
\]

- \(5y_1 + 3y_2 + 4y_3\) will serve as a lower bound for LP objective value if

\[
4y_1 + y_2 \leq 6, \\
2y_1 + y_2 + y_3 \leq 4, \\
y_1 + y_3 \leq 2 \\
y_1, y_2, y_3 \geq 0
\]

- We want as high a lower bound as possible.
LP-Duality

We want to achieve a lower bound on the LP objective value.

**Primal** : minimize \( 6x_1 + 4x_2 + 2x_3 \)
subject to \( 4x_1 + 2x_2 + x_3 \geq 5, \)
\( x_1 + x_2 \geq 3, \)
\( x_2 + x_3 \geq 4, \)
\( x_1, x_2, x_3 \geq 0 \)

**Dual** : maximize \( 5y_1 + 3y_2 + 4y_3 \)
subject to \( 4y_1 + y_2 \leq 6, \)
\( 2y_1 + y_2 + y_3 \leq 4, \)
\( y_1 + y_3 \leq 2 \)
\( y_1, y_2, y_3 \geq 0 \)
LP-Duality

**Primal**: minimize \( \sum_{j=1}^{n} c_j x_j \) \( (P) \)

subject to \( \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = 1, 2, \ldots, m \)

\( x_j \geq 0, j = 1, 2, \ldots, n \)

**Dual**: maximize \( \sum_{i=1}^{m} b_i y_i \) \( (D) \)

subject to \( \sum_{i=1}^{m} a_{ij} y_i \leq c_j, \quad j = 1, 2, \ldots, n \)

\( y_i \geq 0, i = 1, 2, \ldots, m \)
LP-Duality

**Theorem (Weak Duality)**

If $x$ is a feasible solution to the LP $(P)$, and $y$ a feasible solution to the LP $(D)$, then $\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i$.

**Theorem (Strong Duality)**

If the LPs $(P)$ and $(D)$ are feasible, then for any optimal solution $x^*$ to $(P)$ and any optimal solution $y^*$ to $(D)$

$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$

**Theorem (Complementary Slackness)**

Let $\bar{x}$ and $\bar{y}$ be feasible solutions to the LPs $(P)$ and $(D)$, respectively. Then $\bar{x}$ and $\bar{y}$ obey the complementary slackness conditions if and only if they are optimal solutions to their respective LPs.
LP-Duality: Complementary Slackness Condition

- Let $\bar{x}$, $\bar{y}$ be feasible solutions to ($P$) and ($D$) respectively.
- $\bar{x}$ and $\bar{y}$ obey the complementary slackness conditions
  - if $\sum_{i=1}^{m} a_{i,j} \bar{y}_i = c_j$ for each $j$ such that $\bar{x}_j > 0$
  - if $\sum_{j=1}^{n} a_{i,j} \bar{x}_j = b_i$ for each $i$ such that $\bar{y}_i > 0$
- In other words, whenever a primal variable is nonzero, the corresponding dual constraint is tight. Whenever a dual variable is nonzero, the corresponding primal constraint is tight.
The First Primal Dual Algorithm: $f$-approximation for Set Cover

**Primal**: minimize \( \sum_{S \in \mathcal{S}} c_S x_S \) \hspace{1cm} (P)

subject to \( \sum_{S: e \in S} x_S \geq 1, \forall e \in U \)

\( x_S \geq 0, \forall S \in \mathcal{S} \)
The First Primal Dual Algorithm: \( f \)-approximation for Set Cover

**Primal** : minimize \( \sum_{S \in S} c_S x_S \) \hspace{1cm} (P)

subject to \( \sum_{S : e \in S} x_S \geq 1, \forall e \in U \) Dual variable: \( y_e \)

\( x_S \geq 0, \forall S \in S \)

**Dual** : maximize \( \sum_{e \in U} y_e \)

subject to \( \sum_{e : e \in S} y_e \leq c_S \forall S \in S \) \hspace{1cm} (Type-D)

\( y_e \geq 0, \forall e \in U \)
The First Primal Dual Algorithm: $f$-approximation for Set Cover

- Initialize $Y = U$, $F \leftarrow \emptyset$
- While $U$ is not empty
  - Set $y_e = 0$ for all $e \in U$ \textit{start with a feasible dual solution}
  - Start raising all variables $y_e$ such that $e \in U$ until a new Type-D constraint $\sum_{e : e \in S} y_e \leq c_S$ becomes tight (met with equality)
  - Include $S$ in $F$, remove all $e \in S$ from $U$
The First Primal Dual Algorithm: $f$-approximation for Set Cover

- Initialize $Y = U$, $F \leftarrow \emptyset$
- While $U$ is not empty
  - Set $y_e = 0$ for all $e \in U$ *start with a feasible dual solution*
  - Start raising all variables $y_e$ such that $e \in U$ until a new Type-D constraint $\sum_{e : e \in S} y_e \leq c_S$ becomes tight (met with equality)
  - Include $S$ in $F$, remove all $e \in S$ from $U$

Claim

$F$ is a set cover.

Proof.

We discard an element $e$ from $U$ only when we include $S$ in $F$ and $S$ covers $e$. □

Claim

Final solution $\bar{y}$ is a feasible dual solution.
The First Primal Dual Algorithm: \( f \)-approximation for Set Cover

- Initialize \( Y = U, F \leftarrow \phi \)
- While \( U \) is not empty
  - Set \( y_e = 0 \) for all \( e \in U \) **start with a feasible dual solution**
  - Start raising all variables \( y_e \) such that \( e \in U \) until a new Type-D constraint \( \sum_{e: e \in S} y_e \leq c_S \) becomes tight (met with equality)
  - Include \( S \) in \( F \), remove all \( e \in S \) from \( U \)

\[
c(F) = \sum_{S \in F} c(S) \]
\[
= \sum_{S \in F} \sum_{e: e \in S} y_e = \sum_{e \in U} \sum_{S \in F: e \in S} y_e \leq f \sum_{e \in U} y_e \leq f \sum_{e \in U} y_e^* = f \cdot OPT
\]