Cuts & Metrics: Multiway Cut

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Cuts and Metrics
Multiway Cut

Probabilistic Approximation of Metrics by Tree Metric
Cuts & Metrics

Metric Space: A metric \((V, d)\) on a set of vertices \(V\) gives a distance \(d_{uv}\) for each pair of vertices \(u, v \in V\) such that the following properties are obeyed.

- \(d_{u,v} = 0\) iff \(v = u\) (semimetric does not respect this property though \(d(u, u) = 0\) always)
- \(d_{u,v} = d_{v,u}\) \(\forall u, v \in V\)
- \(d_{u,v} \leq d_{u,w} + d_{w,v}\) \(\forall u, v, w \in V\)

May some time ignore the distinction of semimetric and metric

Metrics are useful way of thinking about graph problems involving cuts.
For any cut $S \subseteq V$, define $d_{u,v} = 1$ if $u \in S$ and $v \notin S$, else $d(u,v) = 0$.

If we are trying to find a $s$-$t$ cut which minimizes total weight of edges crossing the cut, we can write:

$$
\min \sum_{e=(u,v) \in E} w_e d_{u,v}
$$

subject to

$$
d_{s,t} = 1
$$
$$
d_{u,v} \leq d_{u,w} + d_{w,v} \quad \forall u, v, w \in V
$$
$$
d_{u,v} = \{0, 1\} \quad \forall u, v \in E
$$
Cut Semimetric

- For any cut $S \subseteq V$, define $d_{u,v} = 1$ if $u \in S$ and $v \notin S$, else $d(u, v) = 0$
- If we are trying to find a $s$-$t$ cut which minimizes total weight of edges crossing the cut, we can write:
  \[
  \min \sum_{e=(u,v) \in E} w_e d_{u,v}
  \]
  subject to
  \[
  d_{s,t} = 1
  \]
  \[
  d_{u,v} \leq d_{u,w} + d_{w,v} \quad \forall u, v, w \in V
  \]
  \[
  d_{u,v} \in [0, 1] \quad \forall u, v \in E \quad \text{(Metric Relaxation)}
  \]
- We can solve $s$-$t$ mincut via Metric Relaxation.
Problem

Given an undirected graph \( G = (V, E) \), costs \( c_e \geq 0 \) for all edges \( e \in E \) and \( k \) distinguished vertices \( s_1, s_2, \ldots, s_k \), remove a minimum-cost collection of edges such that every pair of distinguished vertices \( s_i, s_j, i \neq j \), get separated.
Problem
Given an undirected graph $G = (V, E)$, costs $c_e \geq 0$ for all edges $e \in E$ and $k$ distinguished vertices $s_1, s_2, \ldots, s_k$, remove a minimum-cost collection of edges such that every pair of distinguished vertices $s_i, s_j, i \neq j$, get separated.

Applications Many! Look for them.
Multiway Cut

Isolating cut for s1

Can compute isolating cut optimally
Multiway Cut

- Isolating cut for $s_1$
- Isolating cut for $s_2$
- Isolating cut for $s_3$

Return union of the isolating cuts

- A 2-approximation.
\( \frac{3}{2} \)-approximation for Multiway Cut Problem

- Partition the set of vertices into \( C_1, C_2, ..., C_k \) such that \( s_i \in C_i \), for \( i = 1, 2, .., k \), to minimize the edges crossing them.
- Formulate it in terms of integer linear program.
- Obtain a linear programming relaxation.
- Round (Randomized Rounding) it utilizing connection to metrics.
$\frac{3}{2}$-approximation for Multiway Cut Problem

- $k$ different variables $x^i_u$ for a vertex $u$, $x^i_u = 1$ if $u \in C_i$.
- $k$ different variables $z^i_e$ for an edge $e$, $z^i_e = 1$ if $e \in \delta(C_i)$.
- Each edge in exactly two $C_i, C_j$. Hence the objective is

$$\min \frac{1}{2} \sum_{e \in E} c_e \sum_{i=1}^{k} z^i_e$$

- Constraints:

$$x^i_{s_i} = 1, \hspace{0.5cm} \forall i = 1, 2, .., k$$

$$\sum_{i=1}^{k} x^i_u = 1, \hspace{0.5cm} \forall u \in V$$

For all edge $e = (u, v)$

$$z^i_e \geq |x^i_u - x^i_v|$$

$$x^i_u = \{0, 1\}, z^i_e = \{0, 1\} \hspace{0.5cm} \forall i, u \in V, e \in E$$
\(\frac{3}{2}\)-approximation for Multiway Cut Problem

- \(k\) different variables \(x^i_u\) for a vertex \(u\), \(x^i_u = 1\) if \(u \in C_i\).
- \(k\) different variables \(z^i_e\) for an edge \(e\), \(z^i_e = 1\) if \(e \in \delta(C_i)\).
- Each edge in exactly two \(C_i, C_j\). Hence the objective is

\[
\min \frac{1}{2} \sum_{e \in E} c_e \sum_{i=1}^k z^i_e
\]

- Constraints:

\[
x^i_{s_i} = 1, \quad \forall i = 1, 2, .., k \\
\sum_{i=1}^k x^i_u = 1, \quad \forall u \in V \\
\text{For all edge } e = (u, v) \quad z^i_e = |x^i_u - x^i_v| \\
x^i_u = \{0, 1\}, z^i_e = \{0, 1\} \quad \forall i, u \in V, e \in E
\]
\( \frac{3}{2} \)-approximation for Multiway Cut Problem

- \( k \) different variables \( x_u^i \) for a vertex \( u \), \( x_u^i = 1 \) if \( u \in C_i \).
- \( k \) different variables \( z_e^i \) for an edge \( e \), \( z_e^i = 1 \) if \( e \in \delta(C_i) \).
- Each edge in exactly two \( C_i, C_j \). Hence the objective is

\[
\min \frac{1}{2} \sum_{e=(u,v) \in E} \sum_{i=1}^{k} c_e |x_u^i - x_v^i|
\]

- Constraints:

\[
x_{s_i}^i = 1, \quad \forall i = 1, 2, \ldots, k
\]

\[
\sum_{i=1}^{k} x_u^i = 1, \quad \forall u \in V
\]

\[
x_u^i = \{0, 1\}, \quad \forall i, u \in V
\]
\( \frac{3}{2} \)-approximation for Multiway Cut Problem

- \( k \) different variables \( x^i_u \) for a vertex \( u \), \( x^i_u = 1 \) if \( u \in C_i \).
- \( k \) different variables \( z^i_e \) for an edge \( e \), \( z^i_e = 1 \) if \( e \in \delta(C_i) \).
- Each edge in exactly two \( C_i, C_j \). Hence the objective is

\[
\min \frac{1}{2} \sum_{e=(u,v) \in E} c_e \|x_u - x_v\|_1
\]

- Constraints:

\[
x^i_{s_i} = 1, \quad \forall i = 1, 2, \ldots, k \\
\sum_{i=1}^{k} x^i_u = 1, \quad \forall u \in V \\
x^i_u = \{0, 1\}, \quad \forall i, u \in V
\]
$\frac{3}{2}$-approximation for Multiway Cut Problem

- $k$ different variables $x^i_u$ for a vertex $u$, $x^i_u = 1$ if $u \in C_i$.
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- Each edge in exactly two $C_i, C_j$. Hence the objective is

$$\min \frac{1}{2} \sum_{e=(u,v)\in E} c_e \|x_u - x_v\|_1$$

- Constraints:

$$x^i_{s_i} = 1, \ \forall i = 1, 2, .., k$$

$$\sum_{i=1}^{k} x^i_u = 1, \ \forall u \in V$$

$$x^i_u \in [0, 1] \ \forall i, u \in V (\text{Relaxation})$$
\(\frac{3}{2}\)-approximation for Multiway Cut Problem

- \(k\) different variables \(x_u^i\) for a vertex \(u\), \(x_u^i = 1\) if \(u \in C_i\).
- \(k\) different variables \(z_e^i\) for an edge \(e\), \(z_e^i = 1\) if \(e \in \delta(C_i)\).
- Each edge in exactly two \(C_i, C_j\). Hence the objective is

\[
\min \frac{1}{2} \sum_{e=(u,v) \in E} c_e \|x_u - x_v\|_{l_1}
\]

- Constraints:

\[
\begin{align*}
    x_s^i &= e_i, \quad \forall i = 1, 2, \ldots, k \\
    x_u &\in \Delta_k \quad \forall u \in V (\text{Relaxation})
\end{align*}
\]

- Find location of \(U - \{s_1, s_2, \ldots, s_k\}\) points in \(k\)-dimensional simplex such that their all-pair weighted sum of \(l_1\)-distances is minimized.
Intuition.

- Gives location to points in simplex.
- If a point is closest to $e_i$ assign it to $C_i$
- Use randomization to define ”close” so that each edge $(u, v)$ is cut with probability $\frac{3}{4} ||x_u - x_v||_1$
\( \frac{3}{2} \)-approximation for Multiway Cut Problem

- For any \( r \geq 0 \), define \( B(e_i, r) = \{ u \in V : \frac{1}{2} \| e_i - x_u \|_1 \leq r \} \)
- All vertices are within radius 1.
Algorithm.

- Let $x$ be an optimum LP solution
- for all $1 \leq i \leq k$ do $C_i \leftarrow \phi$
- Pick $r \in (0, 1)$ uniformly at random
- Pick a random permutation $\pi$ of $\{1, 2, ..., k\}$
- $X \leftarrow \phi$ /// $X$ keeps track of vertices already assigned
- for $i = 1$ to $k - 1$ do
  - $C_{\pi(i)} \leftarrow B(s_{\pi(i)}, r) - X$
  - $X \leftarrow X \cup C_{\pi(i)}$
- $C_{\pi(k)} \leftarrow V - X$
- return $F = \bigcup_{i=1}^{k} \delta(C_i)$
\[ \frac{3}{2} \text{-approximation for Multiway Cut Problem} \]

**Analysis**

**Lemma**
\[ |x_u^l - x_v^l| \leq \frac{1}{2}||x_u - x_v||_{l_1} \]

**Lemma**
\[ u \in B(s_i, r) \text{ if and only if } 1 - x_u^i \leq r. \]

- Show that \( \text{Prob}((u, v)\text{is cut}) \leq \frac{3}{4}||x_u - x_v||_{l_1}. \)
\( \frac{3}{2} \)-approximation for Multiway Cut Problem

**Analysis**

- Show that \( \text{Prob}((u, v) \text{ is cut}) \leq \frac{3}{4} \| x_u - x_v \|_{l_1} \).

Define events

\( S_i : i \) is the first in the permutation s.t. one of \( u \) or \( v \) is in \( B(e_i, r) \)

\( X_i : \) exactly one of \( u \) or \( v \) belongs to \( B(e_i, r) \)

For \( e \) to be cut, there must exists at least one \( i \) such that both \( S_i \) and \( X_i \) happens.

Compute \( \sum_{i=1}^{k} \text{Prob}(S_i \cap X_i) \)
3/2-approximation for Multiway Cut Problem

\[
\text{Prob}(X_i) = \text{Prob} \left( \min \left(1 - x'_u, 1 - x'_v \right) \leq r \leq \max \left(1 - x'_u, 1 - x'_v \right) \right) \\
= \left| x'_u - x'_v \right| \leq \frac{1}{2} \left| x_u - x_v \right|_{L_1}
\]

Let \( l \) be the vertex which minimizes \( \min_i \min(1 - x'_u, 1 - x'_v) \). If \( i \neq l \) and \( l \) comes before \( i \) in \( \pi \) then \( S_i \) is false.

\[
\text{Prob}(S_i \cap X_i) \\
= \text{Prob}(S_i \cap X_i | \pi(l) < \pi(i)) \cdot \text{Prob}(\pi(l) < \pi(i)) \\
+ \text{Prob}(S_i \cap X_i | \pi(l) > \pi(i)) \cdot \text{Prob}(\pi(l) > \pi(i)) \\
= \text{Prob}(S_i \cap X_i | \pi(l) > \pi(i)) \cdot \text{Prob}(\pi(l) > \pi(i)) \\
= \frac{1}{2} \text{Prob}(S_i \cap X_i | \pi(l) > \pi(i)) \\
\leq \text{Prob}(X_i | \pi(l) > \pi(i)) \cdot \frac{1}{2} = \text{Prob}(X_i) \cdot \frac{1}{2} = \frac{1}{2} \left| x'_u - x'_v \right|
\]
\[ \frac{3}{2} \text{-approximation for Multiway Cut Problem} \]

\[ \text{Prob}(S_i \cap X_i) = \text{Prob}(X_i) \frac{1}{2} = \frac{1}{2} |x_u^i - x_v^i| \]

\[ \sum_i \text{Prob}(S_i \cap X_i) = \sum_{i, i \neq l} \frac{1}{2} |x_u^i - x_v^i| + \text{Prob}(S_l \cap X_l) \]

\[ \leq \frac{1}{2} (||x_u - x_v|| - (|x_u^l - x_v^l|)) + |x_u^l - x_v^l| = \frac{1}{2} (||x_u - x_v|| + |x_u^l - x_v^l|) \]

\[ \leq \frac{3}{2} \cdot \frac{3}{2} ||x_u - x_v|| = \frac{3}{4} ||x_u - x_v|| \]
Use tree metrics to approximate a given distance metric \( d \) on a set of vertices \( V \).

A tree metric \((V', T)\) for a set of vertices \( V \) is a tree \( T \) on \( V' \supseteq V \) with nonnegative edge lengths. Distance \( T_{u,v} \) is the unique shortest path length from \( u \) to \( v \).

We want for some small \( \alpha \geq 1 \) the following to hold for all vertices \( u, v \in V \):

\[
d(u, v) \leq T(u, v) \leq \alpha d(u, v)
\]

\( \alpha \): distortion of embedding.
Importance of Low Distortion Embedding

- Many problems are easy to solve on trees.
- Use low distortion embedding to reduce problem on general metric space to tree metric.
- $\alpha$ dictates the approximation factor.
- See section 8.6 of Shmoys-Williamsons for examples.
Tree Embedding Theorem We Want to Prove

Unfortunately no deterministic embedding can have distortion $o(n)$. Embed into a distribution of trees. Show expected distortion is low.

**Theorem**

*Given a distance metric $(V, d)$, such that $d(u, v) \geq 1$ for all $u \neq v$, $u, v \in V$, there is a randomized, polynomial-time algorithm that produces a tree metric $(V', T)$, $V \subseteq V'$, such that for all $u, v \in V$, $d(u, v) \leq T(u, v)$ and $E[T(u, v)] \leq O(\log n)d(u, v)$.*
Hierarchical Cut Decomposition

Let $\Delta = 2^k$ such that $2^k \geq 2 \max_{u,v} \{d(u,v)\} > 2^{k-1}$.
Hierarchical Cut Decomposition

Lemma
For all pairs of vertices $u, v \in V$, $d(u, v) \leq T(u, v)$.

Proof.

- Let $d(u, v) = 2^k + r$, $2^k < r < 0$. $u$ and $v$ cannot appear together in any set at level $k - 1$ or lower. Hence the lowest level in which $u$ and $v$ may appear together is $k$. The total distance is

$$\geq 2 \sum_{i=1}^{k} 2^i = 4(1 + 2 + 2^2 + ... + 2^{k-1}) \geq 2^{k+2} - 4 \geq d(u, v).$$

- If $d(u, v) = 2^k$, then the lowest level in which $u$ and $v$ may appear together is $k - 1$, $k \geq 2$. The total distance is

$$\geq 2 \sum_{i=1}^{k-1} 2^i = 4(1 + 2 + 2^2 + ... + 2^{k-2}) \geq 2^{k+1} - 4 \geq d(u, v).$$
Hierarchical Cut Decomposition

Lemma

If the least common ancestor of vertices \( u, v \in V \) is at level \( i \) then \( T(u, v) \leq 2^{i+2} \).
Hierarchical Cut Decomposition

- Pick a random permutation \( \pi \) of \( V \)
- Set \( \Delta \) to the smallest power of 2 greater than \( 2 \max_{u,v} d(u, v) \)
- Pick \( r_0 \in [\frac{1}{2}, 1) \) uniformly at random; set \( r_i = 2^i r_0 \) for all \( 1 \leq i \leq \log_2 \Delta \)
- \( \text{// } C(i) \) will be the sets corresponding to the nodes at level \( i \), it creates a partition of \( V \).
- \( C(\log_2 \Delta) = \{ V \} \): create tree node corresponding to \( V \).
Hierarchical Cut Decomposition

\[ \text{for } i \leftarrow \log_2 \Delta \text{ downto } 1 \text{ do} \]
\[ C(i - 1) \leftarrow \emptyset \]
\[ \text{for all } C \in C(i) \text{ do} \]
\[ S \leftarrow C \]
\[ \text{for } j \leftarrow 1 \text{ to } n \text{ do} \]
\[ \text{if } B(\pi(j), r_{i-1}) \cap S \neq \emptyset \text{ then} \]
\[ \quad \text{Add } B(\pi(j), r_{i-1}) \cap S \text{ to } C(i - 1) \]
\[ \quad \text{Remove } B(\pi(j), r_{i-1}) \cap S \text{ from } S \]
\[ \text{Create tree nodes corresponding to all sets in } C(i - 1), \text{ join them to } C \text{ with edges of length } 2^i \]
Hierarchical Cut Decomposition

- Look at level $i$ and a pair of vertices $u$, $v$
- $S_{iw}$: In the $i$th level $w$ is the first vertex such that at least one of $u$ and $v$ is contained in its ball.
- $X_{iw}$: In the $i$th level exactly one of $u$ and $v$ is contained in the ball of $w$

\[
T(u, v) \leq \max_{i=0, \ldots, \log \Delta - 1} 1(\exists w \in V : X_{i,w} \cap S_{i,w}).2^{i+3}
\]

\[
\leq \sum_{w \in V} \sum_{i=0}^{\log \Delta - 1} 1(X_{i,w} \cap S_{i,w}).2^{i+3}
\]

\[
E[T(u, v)] \leq \sum_{w \in V} \sum_{i=0}^{\log \Delta - 1} \text{Prob}(X_{i,w} \cap S_{i,w}).2^{i+3}
\]
Hierarchical Cut Decomposition

\[ E[T(u, v)] \leq \sum_{w \in V} \sum_{i=0}^{\log \Delta - 1} \text{Prob}(X_{i,w} \cap S_{i,w}) \cdot 2^{i+3} \]

\[ = \sum_{w \in V} \sum_{i=0}^{\log \Delta - 1} \text{Prob}(S_{i,w} | X_{i,w}) \cdot \text{Prob}(X_{i,w}) \cdot 2^{i+3} \]

\[ \leq \sum_{w \in V} b_w \sum_{i=0}^{\log \Delta - 1} \text{Prob}(X_{i,w}) \cdot 2^{i+3} \]
Hierarchical Cut Decomposition

- Show $\sum_{i=1}^{\log \Delta} \text{Prob}(X_{iw}).2^{i+3} \leq 16d_{u,v}$.
- Show if $w$ is the $j$th closest vertex according to \( \min(d(u, w), d(v, w)) \) then $\text{Prob}(S_{i,w}\mid X_{i,w}) \leq \frac{1}{j}$ Set $b_w = \frac{1}{j}$.