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Lecture 15

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1 Group Testing

1.1 Explicit Construction

We want to construct an $m \times n$ t-disjunct testing matrix with number of rows m as close to $t^2 \log n$ as possible. We do that by constructing a constant weight code and then listing the codewords as columns of the matrix. We can then use the simple formula from the last class to conclude about its disjunctive-ness. The following construction is due to Kautz and Singleton.

We construct a concatenated code where the outer code is a q-ary RS code with length q and the inner code is an indicator code that maps each q-ary symbol to a q-length vector.

$$
\begin{aligned}
0 &\longrightarrow 1000...0 \\
1 &\longrightarrow 0100...0 \\
2 &\longrightarrow 0010...0 \\
&. \\
&. \\
&. \\
q &\longrightarrow 0000...1
\end{aligned}
$$

The length of the overall concatenated code is q^2. (The outer code is q-ary and the inner code replaces each q-ary symbol with a q-length vector). The minimum distance is:

$$d = 2(q - \log_q N + 1),$$

Where N is the number of codewords. The weight of every codeword of this binary code is q because the length of outer code is q and inner code has constant weight 1. We construct an $m \times n$ matrix by placing each codeword as column, where $n = N$ and $m = q^2$.

Then:

$$
\begin{aligned}
t &= \frac{q}{q - \frac{d}{2}} = \frac{q}{q - (q - \log_q N + 1)} = \frac{q}{\log_q N - 1} \\
&= \frac{\sqrt{m}}{\log_q N - 1} = \frac{\sqrt{m}}{\frac{\log N}{\log q} - 1} = \frac{\sqrt{m}}{\frac{\log N}{\log \sqrt{m}} - 1} \\
&= \frac{\frac{1}{2}\sqrt{m}\log m}{\log N - \frac{1}{2}\log m} \geq \frac{\frac{1}{2}\sqrt{m}\log m}{\log N} \\
\implies m &\leq \frac{4t^2 \log^2 N}{\log^2 m}
\end{aligned}
$$

We, then, have $m = O(t^2 \log^2 N)$ (only a logarithmic factor away from optimal). This result was found in 1964. In 2008, an explicit construction was found such that $m = O(t^2 \log N)$. We can also design a matrix that is "almost disjunct" (i.e. disjunct for most columns). This works for elements that are independently defective. In this case we can get $m = \frac{t \log^2 n}{\log t}$ (Mazumdar 2016).

2 Compressed Sensing

2.1 Compressed Sensing Problem

When cameras take a picture (or other devices capture data), the sensors capture all of the available pixels (data) and later when the images are stored, they are compressed. During this process, most of the data is thrown away (because it is deemed unnecessary). An alternative to this approach is called Sensed Compressed Signals (a.k.a. Compressed Signaling) (e.g. Single Pixel Camera). In this case there are m sensors and n pixels (data points), where $m << n$. The sensors sense a linear combination of pixels instead of all of the pixels individually. For the Single Pixel Camera, this is done by using mirrors. Each of the m sensors uses some subset of the n pixels (each one belonging to the set of reals: $\mathbb{R}^n$). Each sensor, then takes a linear combination of each of the pixels it has sensed. The result is an image that has already been compressed.

2.2 Explicit Construction

Each signal $x \in \mathbb{R}^n$. We can construct a matrix $A : m \times n$. The compression ratio will be $\frac{m}{n}$. We have the equation $Ax = b$, where b is the compressed image. It is possible to construct x from b, however, we have an underdetermined set of equations, so we cannot uniquely determine x from b in general.

Real world data is usually sparse. We can exploit that sparsity to extract x uniquely from b. Even if the input itself is not sparse, it can be transformed into another domain and become sparse. For example, image data is not sparse when viewed at the pixel level. However, after the Fourier transform, the high frequency part is usually sparse. In general, if Φx is equal to some vector of k non-zero values (where Φ is an $n \times n$ matrix), then the signal is a k-sparse signal. We have $B\Phi x = b$, where $A = B\Phi$ and Φx is a sparse signal. Then we can assume we are doing encoding on sparse Φx and design B.

Then we can define the problem formally:

$$\begin{aligned} &Ax = b \\ &x \in \mathbb{R}^n \\ &||x||_{l_0} \leq k \text{ (k-sparse)} \end{aligned}$$

We need to design A so that we can recover x uniquely from b. Note this definition looks similar to the syndrome decoding where we have a syndrome and want to recover the error uniquely. The difference is that now we are doing this in real domain, and the rows of the standard array are not finite.

To recover x uniquely, Ax_1 should be different from Ax_2 for all x_1, x_2 that are k-sparse, which implies $A(x_1 - x_2) \neq 0$. $x_1 - x_2$ is a $2k$-sparse signal. This implies the linear combination of any $2k$ columns of A is not 0, so any $2k$ columns of A are linearly independent. We can get unique recovery if and only if, any $2k$ columns are linearly independent.

We can construct an $m \times n$ matrix A as follows:

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \dots & a_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{m-1} & a_2^{m-1} & a_3^{m-1} & \dots & a_n^{m-1} \end{bmatrix}$$

We can pick any m columns to form a Vandermonde matrix. The determinant of it is: $\prod_{i \neq j}(a_i - a_j) \neq 0$. This means any m columns in A are linearly independent. Therefore, $m = 2k$ samples will suffice. Note that if x contains noise, this argument fails.

There is still one problem. The real world signal is not precisely spare. It contains many small non-zero values. In this case, the argument above fails.

Therefore we need to relax the definition and allow small error in reconstruction. In general, we can start with any signal $x \in \mathbb{R}^n$ and take $Ax = b$, where A is $m \times n$. x_k is a vector with the k largest values of x (i.e. the best k-sparse approximation). We need to give an estimate $\hat{x}$ based on b such that $||x - \hat{x}|| \leq c||x - x_k||$

Property: A matrix A has the δ_k-RIP (Restricted Isometric Property) if for every k-sparse vector z: $(1 - \delta_k)||z||_{l_2}^2 \leq ||Az||_{l_2}^2 \leq (1 + \delta_k)||z||_{l_2}^2$

Theorem 1 *(Candes-Tao): If A is δ_{2k}-RIP with $\delta_{2k} \leq \sqrt{2} - 1$, then the following algorithm (Basis Pursuit) will produce $\hat{x}$ such that $||x - \hat{x}||_{l_2} \leq \frac{c}{\sqrt{k}}||x - x_k||_{l_1}$*

Basis Pursuit Algorithm: Find the $min||x||_{l_1}$ such that $Ax = b$ using linear programming (can be solved in polynomial time).