COMPSCI 650 Applied Information Theory	April 7 2016
Lecture 19	
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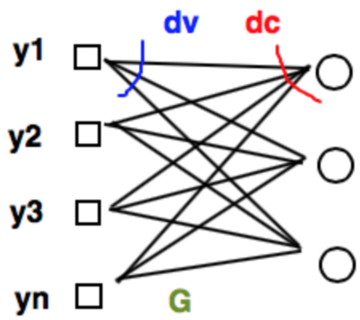
1 Iterative Decoding

Given a possibly corrupted encoding c of input message y, we would like to determine the best estimate of y. Recall that the encoding can be generated using the $k \times n$ (Hamming) generator matrix G and errors can be corrected using the $n - k \times n$ (Hamming) parity check matrix H.



We can also infer the most likely message using Maximum A Priori estimation (MAP) or Maximum Likelihood Estimation (ML or MLE). MAP is done by maximizing the probability of the code, given the message:

$$\max_{c \in C} P(c|y)$$

whereas ML is done by maximizing the likelihood of the code:

$$\max_{c \in C} P(y|c) = \max_{c \in C} P(y|c) \prod_{i=1}^{n} P(y_i|c_i)$$

In the binary case for a single bit, this becomes:

$$\begin{aligned} \arg_{x \in \{0,1\}} \max P(c_i = x|y) &= \arg_{x \in \{0,1\}} \max \sum_{c \in C} P(c|y) \\ &= \arg_{x \in \{0,1\}} \max \sum_{c \in C} P(y|c) \\ &= \arg_{x \in \{0,1\}} \max \sum_{c \in C} \prod_{i=1}^{n} P(y_i|c_i) \end{aligned}$$

We denote the variable degree as d_v and the check degree as d_c. Then the transmission rate is $\frac{n \cdot dv}{dc}$ where n in the number of bits. This should be clear from the image shown above.

The Iterative Decoding (or Message Passing) algorithm is a two step iterative process used to find the best decoding of the encoded message received. In the first step, called the *variable rule*, we compute the log-likelihood ratio as follows:

$$
\begin{aligned}
m &= \log \frac{P(x=0|y_1...y_d)}{P(x=1|y_1...y_d)} \\
&= \log \frac{P(y_1...y_d|x=0)P(x=0)P(y1...y_d)}{P(y_1...y_d|x=1)P(x=1)P(y1...y_d)}
\end{aligned}
$$

Since the priors are the same for both 0 and 1, we can cancel $P(x=0)$ with $P(x=1)$. Then

$$
\begin{aligned}
m &= \log \frac{P(y_1...y_d|x=0)}{P(y_1...y_d|x=1)} \\
&= \log \frac{\prod_{i=1}^{d} P(y_i|x_i=0)}{\prod_{i=1}^{d} P(y_i|x_i=1)}
\end{aligned}
$$

The second equivalence follows from the independence assumption of the memoryless channel. Then, simplifying the log of products to a sum of logs, we get

$$
\begin{aligned}
m &= \sum_{i=1}^{d} \log \frac{P(y_i|x_i=0)}{P(y_i|x_i=1)} \\
&= \sum_{i=1}^{d} l_i
\end{aligned}
$$

where l_i is the log-likelihood ratio for a single bit.

For the second step of the Iterative Decoding algorithm, the *check rule*, we have $d = d_c$ and the log-likelihood ratio is

$$
m = \log \frac{P(x=0|y_1...y_{d-1})}{P(x=1|y_1...y_{d-1})}
$$

Notice the change of the bounds on y. Exponentiating the log-likelihood ratio gives

$$
e^m = \frac{P(x=0|y_1...y_{d-1})}{P(x=1|y_1...y_{d-1})}
$$

Then it follows that

$$
\frac{e^m - 1}{e^m + 1} = \frac{\frac{P(x=0|y_1...y_{d-1})}{P(x=1|y_1...y_{d-1})} - 1}{\frac{P(x=0|y_1...y_{d-1})}{P(x=1|y_1...y_{d-1})} + 1}
$$

Multiplying both numerator and denominator by $P(x=1|y_1...y_{d-1})$ gives

$$
\frac{e^m - 1}{e^m + 1} = \frac{P(x=0|y_1...y_{d-1}) - P(x=1|y_1...y_{d-1})}{P(x=1|y_1...y_{d-1}) + P(x=0|y_1...y_{d-1})}
$$

The denominator evaluates to 1, because it the sum of a distribution over the entire outcome space.

$$\begin{aligned}\frac{e^m-1}{e^m+1} &= P(x=0|y_1...y_{d-1}) - P(x=1|y_1...y_{d-1}) \\ &= P(x_1+x_2+...+x_{d-1}=0|y_1...y_{d-1}) - P(x_1+x_2+...+x_{d-1}=1|y_1...y_{d-1}) \\ &= \prod_{i=1}^{d-1}\left(P(x_i=0|y_i) - P(x_i=1|y_i)\right)\end{aligned}$$

For a single bit, the log-likelihood ratio is:

$$\begin{aligned}l_i &= \log\frac{P(y_i|x_i=0)}{P(y_i|x_i=1)} \\ &= \log\frac{P(x_i=0|y_i)}{P(x_i=1|y_i)}\end{aligned}$$

Then the ratio

$$\begin{aligned}\frac{e^m-1}{e^m+1} &= P(x_i=0|y_i) - P(x_i=1|y_i) \\ &= \prod_{i=1}^{d-1}\frac{e^{l_i}-1}{e^{l_i}+1} \\ &= \prod_{i=1}^{d-1}\frac{e^{\frac{l_i}{2}}-e^{\frac{l_i}{2}}}{e^{\frac{l_i}{2}}+e^{\frac{l_i}{2}}} \\ &= \prod_{i=1}^{d-1}\tanh(\frac{l_i}{2})\end{aligned}$$

Notice that

$$(e^m-1)+(e^m+1)=2e^m$$

Dividing everything by e^m+1 gives the equality:

$$\frac{e^m-1}{e^m+1}+1=\frac{2e^m}{e^m+1}$$

Likewise, notice that

$$(e^m+1)-(e^m-1)=2$$

Dividing by e^m+1 gives:

$$1-\frac{e^m-1}{e^m+1}=\frac{2}{e^m+1}$$

Then it follows that

$$e^m = \frac{1 + \prod_{i=1}^{d-1} \tanh(\frac{l_i}{2})}{1 - \prod_{i=1}^{d-1} \tanh(\frac{l_i}{2})}$$

which implies

$$m = \log \frac{1 + \prod_{i=1}^{d-1} \tanh(\frac{l_i}{2})}{1 - \prod_{i=1}^{d-1} \tanh(\frac{l_i}{2})}$$

1.1 Implementation

The algorithm follows directly from the steps above. In each iteration of the message passage algorithm, do the following:

1. Initialization
2. Variable Rule
3. Check Rule

2 Expander Graphs

An expander graph is a strongly connected sparse graph. We can define an expander graph over the encoding as follows

$$G = \{S \in V_1 : |S| \leq \alpha n, |N(S)| \geq \gamma d_v |S|\}$$

for some positive α and γ, where V_1 is the set of nodes in the message prior to encoding and $N(S)$ is the set of neighbors of S.

In the next class we will show that if $\gamma > \frac{3}{4}$, we will be able to correct up to $\frac{\alpha}{2} n = \mathcal{O}(n)$ errors.