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Lecture 11

Instructor: Arya Mazumdar *Scribe: Names Redacted*

Let X and X' denote two iid random variable. Then, we have:

$$\Pr(X = X') \geq 2^{-H(X)} \tag{1}$$

where $H(\cdot)$ denotes the entropy of a random variable.
Proof

$$\Pr(X = X') = \sum_{a \in X} \Pr(X = a | X' = a) \Pr(X' = a) = \sum_{a \in X} (p(a))^2 = 2^{\log \sum (p(a))^2} \tag{2}$$

According to the Jensen inequality, since logarithm is a concave function, then we have:

$$\log \sum (p(a))^2 = \log \mathbb{E}[p(a)] \geq \mathbb{E}[\log p(a)] = \sum_c p(c) \log p(c) \tag{3}$$

We can then re-write Equation (3) using Equation (2) as follows:

$$\Pr(X = X') \geq 2^{\sum_c p(c) \log p(c)} = 2^{-H(X)} \tag{4}$$

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1 MAB: Multi-Armed Bandit

Let a_t be the action of an agent at time t. The regret function $R(N, T)$ for T time and N actions is computed as follows:

$$R(N, T) = \mathbb{E}\left[\sum_{t=1}^{T} l(a_t) - \min_{i \in \{1, \ldots, N\}} l(i)\right] \tag{5}$$

where l denotes the loss function of each action. It can be shown that $R(N, T) \geq c\sqrt{NT}$ where c is a constant number. There is a generic algorithm for MAB with the regret value of $O(\sqrt{NT \log N})$.

In the following subsections we prove that $R(N, T) \geq c\sqrt{NT}$.

1.1 Identifying Whether a Coin is Biased

Lemma 1 *We need $O(\frac{1}{\epsilon^2})$ coin tosses to discern a biased coin with the probability of $\frac{1}{2} - \epsilon$ for head (and obviously with the probability of $\frac{1}{2} + \epsilon$ for tail), from an unbiased coin.*

Proof Consider two hypothesises H1 and H2 where respectively denote biased and unbiased coins. In other words, we have the following hypothesises:

$$\begin{cases} H1 : biased & p(h) = \frac{1}{2} - \epsilon \\ H2 : unbiased & p(h) = \frac{1}{2} \end{cases} \tag{6}$$

According to the Le Cam's identity, we can compute the error probability as follows:

$$P_e = \frac{1}{2}\left[1 - \|p_1^{(m)} - p_2^{(m)}\|_{TV}\right] \tag{7}$$

where m denotes the number of coin tosses.

Using the Pinsker's Inequality we can find an upper-bound for the total variation distance between two probabilistic distribution as follows:

$$\|p_1^{(m)} - p_2^{(m)}\|_{TV}^2 \leq \frac{2}{\ln 2} D(p_1^{(m)} || p_2^{(m)}) \tag{8}$$

Considering Equation (7) and Equation (8), we have:

$$P_e \geq \frac{1}{2}\left[1 - \sqrt{(2/\ln 2) D(p_1^{(m)} || p_2^{(m)})}\right] \tag{9}$$

We now need to introduce a nice property of the KL-divergence as follows:

Proposition 2 *Suppose $P(X,Y)$ and $Q(X,Y)$ two joint distributions. If X and Y denote two independent random variables, then:*

$$D(P(X,Y)||Q(X,Y)) = D(P(X)||Q(X)) + D(P(Y)||Q(Y)) \tag{10}$$

Proof

$$\begin{aligned} D(P(X,Y)||Q(X,Y)) &= \sum_{x,y} P(x,y) \log \frac{P(x,y)}{Q(x,y)} \\ &= \sum_{x,y} P(x)P(y) \log \frac{P(x)P(y)}{Q(x)Q(y)} \\ &= \sum_{x} P(x) \log \frac{P(x)}{Q(x)} + \sum_{y} P(y) \log \frac{P(y)}{Q(y)} \\ &= D(P(X)||Q(X)) + D(P(Y)||Q(Y)) \end{aligned} \tag{11}$$

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Considering the result of Proposition 2, we can re-write Equation (9) as below:

$$P_e \geq \frac{1}{2}\left[1 - \sqrt{(2m/\ln 2) D(p_1 || p_2)}\right] \tag{12}$$

Keep in mind that coin tosses are independent of each other and $p^{(m)}$ denotes the probability of m coin tosses. We can compute $D(p_1||p_2)$ as follows:

$$\begin{aligned} D(p_1||p_2) = D(\frac{1}{2} - \epsilon || \frac{1}{2}) &= (\frac{1}{2} - \epsilon) \log(\frac{\frac{1}{2} - \epsilon}{\frac{1}{2}}) + (\frac{1}{2} + \epsilon) \log(\frac{\frac{1}{2} + \epsilon}{\frac{1}{2}}) \\ &= \underbrace{\frac{1}{2} \log((1-2\epsilon)(1+2\epsilon))}_{negative} + \epsilon \log(\frac{1+2\epsilon}{1-2\epsilon}) \\ &\leq \epsilon \log(1 + \frac{4\epsilon}{1-2\epsilon}) \\ &\leq \frac{4\epsilon^2}{1-2\epsilon} \qquad\qquad (because 1 + x \leq e^x) \\ &\leq 8\epsilon^2 \qquad\qquad (\forall \epsilon \leq 1/4) \end{aligned} \tag{13}$$

Considering both Equation (12) and Equation (13), we have:

$$P_e \geq \frac{1}{2}\left[1 - \sqrt{(2m/\ln 2) 8\epsilon^2}\right] = \frac{1}{2}\left[1 - \sqrt{(16/\ln 2)\epsilon^2 m}\right] \tag{14}$$

Suppose $P_e < \frac{1}{4}$. Then:

$$\frac{1}{4} > \frac{1}{2}\left[1 - \sqrt{(16/\ln 2)\epsilon^2 m}\right] \Rightarrow \sqrt{(16/\ln 2)\epsilon^2 m} > \frac{1}{2} \Rightarrow \frac{16}{\ln 2}\epsilon^2 m > \frac{1}{4} \Rightarrow m > \frac{\ln 2}{64\epsilon^2} \tag{15}$$

Therefore, we need at least $\frac{\ln 2}{64\epsilon^2}$ coin tosses to have lower than $\frac{1}{4}$ error probability. We show that we need $O(\frac{1}{\epsilon^2})$ coin tosses and thus, the proof of Lemma 1 is completed (we can easily put parameters instead of $P_e < \frac{1}{4}$).
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1.2 Identifying A Biased Coin

In the previous subsection, we show that we need $O(\frac{1}{\epsilon^2})$ coin tosses to decide whether a coin is biased or not. Now, assume that we have N coins and our purpose is to decide which coin is biased.

Lemma 3 *At least $O(\frac{N}{\epsilon^2})$ coin tosses needed to decide which coin among N coins are biased.*

In other words, suppose (a_t, b_t) is a pair in which $a_t \in \{1, 2, ..., N\}$ and $b_t \in \{1, 2, ..., N\}$ respectively denote the action (choose one coin and toss) and the decision (decide which coin is biased) at time t. If $t < \frac{N}{100\epsilon^2}$ then there exists a set $J \subset \{1, 2, ..., N\}$ and $|J| \geq N/3$ where $\forall j \in J : \Pr(b_t = j) < 1/2$. In fact, we cannot decide which coin is biased with a high probability. We will prove it.
Proof Let A_j be the number of times that ALG (a method for deciding which coin is biased) tosses the j^{th} coin. It is obvious that:

$$\sum_{i=1}^{N} A_j = t \tag{16}$$

where t denotes the total number of coin tosses. Consider $J_1 = \{j \in \{1, ..., N\} : A_j \leq \frac{3t}{N}\}$. Therefore, $N - |J_1|$ denotes the number coins that are tossed at least $\frac{3t}{N}$ times. Thus, we have:

$$(N - |J_1|)\frac{3t}{N} \leq t \Rightarrow N - |J_1| = \frac{N}{3} \Rightarrow |J_1| \geq \frac{2N}{3} \tag{17}$$

Now, consider $J_2 = \{j \in \{1, ..., N\} : \Pr(b_t = j) \leq \frac{3}{N}\}$. Since, the sum of all probabilities should be equal to one, thus we have:

$$(N - |J_2|)\frac{3}{N} \leq 1 \Rightarrow |J_2| \geq \frac{2N}{3} \tag{18}$$

It is obvious that $N \geq |J_1 \cup J_2|$ and we know that

$$|J_1 \cup J_2| = |J_1| + |J_2| - |J_1 \cap J_2| \tag{19}$$

According to Equation (17) and Equation (18), we have:

$$|J_1 \cup J_2| \geq \frac{2N}{3} + \frac{2N}{3} - |J_1 \cap J_2| \Rightarrow |J_1 \cap J_2| \geq \frac{N}{3} \tag{20}$$

Now, consider $J \equiv J_1 \cap J_2$. Then, according to the Le Cam's Inequality we have:

$$\begin{aligned} P_{H1}(b_t = j) &\leq P_{H2}(b_t = j) + \|P_{H1}(b_t = j) - P_{H2}(b_t = j)\| \\ &\leq \frac{3}{N} + \sqrt{\frac{2}{\ln 2} D(P_{H1}^{A_j} || P_{H2}^{A_j})} \\ &= \underbrace{\frac{3}{N}}_{\to 0} + \sqrt{\frac{2}{\ln 2} \underbrace{A_j}_{\frac{3t}{N}} \underbrace{D(\frac{1}{2} - \epsilon || \frac{1}{2})}_{\leq 8\epsilon^2}} \\ &\simeq \sqrt{\frac{t\epsilon}{N} . \frac{16}{\ln 2}} \end{aligned} \tag{21}$$

In the above equations, we use the results of Proposition 2 and Equation 13. Now, suppose $t < \frac{N \ln 2}{64\epsilon^2}$, then $P_{H1}(b_t = j) < \frac{1}{2}$. So this is also true for $t \leq \frac{N}{100\epsilon^2}$. So Lemma 3 is successfully proved.
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1.3 Proof of the Regret Lower-Bound in MAB

As pointed out in the beginning of Section 1, the regret function is calculated as:

$$R(N,T) = \mathbb{E}\left[\sum_{t=1}^{T} l(a_t) - \min_{i\in\{1,\ldots,N\}} l(i)\right] \tag{22}$$

Consider hypothesis H1 as:

$$l(i) = \begin{cases} 1 & w/prob. \frac{1}{2} \\ 2 & w/prob. \frac{1}{2} \end{cases} \tag{23}$$

Also consider hypothesis H2 as:

$$l(i) = \begin{cases} 1 & w/prob. \frac{1}{2} - \epsilon \\ 2 & w/prob. \frac{1}{2} + \epsilon \end{cases} \tag{24}$$

According to these two hypothesis we have:

$$\sum_{i=1}^{T} \mathbb{E}\left[\min_{i\in\{1,\ldots,N\}} l(i)\right] = (\frac{1}{2} - \epsilon)T \tag{25}$$

The other term in the regret function can be also computed as below (assume that J^* is the biased coin):

$$\begin{aligned} \mathbb{E}[l(a_t)] &= \mathbb{E}[l(a_t)|j^* \in J] \Pr(j^* \in J) + \mathbb{E}[l(a_t)|j^* \notin J] \Pr(j^* \notin J) \\ &= \frac{1}{3}\mathbb{E}[l(a_t)|j^* \in J] \Pr(j^* \in J) + \frac{2}{3}\mathbb{E}[l(a_t)|j^* \notin J] \Pr(j^* \notin J) \\ &\geq \frac{1}{3}[\frac{1}{2}(\frac{1}{2} - \epsilon) + \frac{1}{2}/\frac{1}{2}] + \frac{2}{3}[\frac{1}{2} - \epsilon] \\ &= \frac{1}{3}[\frac{1}{2} - \frac{\epsilon}{2}] + \frac{2}{3}[\frac{1}{2} - \epsilon] \\ &= \frac{1}{2} - \frac{2}{3}\epsilon \end{aligned} \tag{26}$$

According to Equation (25) and Equation (26), the regret function can be calculated as:

$$R(N,T) = (\frac{1}{2} - \frac{2}{3}\epsilon)T - (\frac{1}{2} - \epsilon)T = \frac{1}{3}\epsilon T \tag{27}$$

If we tosses more than $\frac{N}{100\epsilon^2}$ times, we have $\epsilon \leq \frac{1}{10}\sqrt{\frac{N}{T}}$. Therefore, the regret is equal to:

$$R(N,T) \geq \frac{1}{30}\sqrt{NT} \tag{28}$$

Now, the proof is completed.

References

- `http://www.cs.cornell.edu/courses/cs683/2007sp/lecnotes/week8.pdf`
- `http://ttic.uchicago.edu/ madhurt/courses/infotheory2014/l6.pdf`