COMPSCI 650: HOMEWORK 3

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All problems carry equal points
Some problems are from Cover and Thomas, 2nd Ed.
Due Date: Apr 19, 2016: Start of Class

(1) (Fisher Information) Let $f(x;\theta) = \theta e^{-\theta x}, x \geq 0$ be the probability distribution of a random variable X.
- Compute the score function for f.
- Compute the Fisher information for f, θ. What is the Cramer-Rao bound on the unbiased estimator?
- Suppose we observe iid samples of X, namely $X_1, \dots X_n$. What would be the Cramer-Rao bound on the unbiased estimator of θ from this n samples?
- Design an estimator for θ from $X_1, \dots X_n$. Does your estimator match the Cramer-Rao bound?

(2) (Channel Capacity) Consider the channel where the input alphabet $\mathcal{X} = \{0, 1\}$, output alphabet $\mathcal{Y} = \{0, 1, a, 1 + a\}$. The input-output transition probabilities are given by,

$$p(y|x) = \begin{cases} \frac{1}{2}, & \text{if } y = x; \\ \frac{1}{2}, & \text{if } y = x + a; \\ 0, & \text{otherwise.} \end{cases}$$

What is the capacity of this channel?

(3) (Channel Capacity) Consider the channel where the input and output alphabets are same, $\mathcal{X} = \mathcal{Y} = \{0, 1\}$. The channel transition probabilities are given by,

$$p(y|x) = \begin{cases} 1, & \text{if } y = x = 0; \\ p, & \text{if } x = 1, y = 0; \\ 1 - p, & \text{if } x = 1, y = 1; \\ 0, & \text{otherwise.} \end{cases}$$

What is the capacity of this channel?

(4) Consider a commander of an army besieged in a fort for whom the only means of communication to his allies is a set of carrier pigeons. Assume that each carrier pigeon can carry one letter (8 bits), that pigeons are released once every 5 minutes, and that each pigeon takes exactly 3 minutes to reach its destination.
- Assuming that all the pigeons reach safely, what is the capacity of this link in bits/hour?
- Now assume that the enemies try to shoot down the pigeons and that they manage to hit a fraction α of them. Since the pigeons are sent at a constant rate, the receiver knows when the pigeons are missing. What is the capacity of this link?

- Now assume that the enemy is more cunning and that every time they shoot down a pigeon, they send out a dummy pigeon carrying a random letter (chosen uniformly from all 8-bit letters). What is the capacity of this link in bits/hour?

Set up an appropriate model for the channel in each of the above cases, and indicate how to go about finding the capacity.

(5) (Exponential noise channels.) Let the output Y_i and input X_i of a channel is related by $Y_i = X_i + Z_i$, where Z_i is i.i.d. exponentially distributed noise with mean μ. Assume that we have a mean constraint on the signal (i.e., $EX_i \leq \lambda$). What is the capacity of such a channel?

(6) Consider the binary Hamming code of length 15, i.e., the code whose parity-check matrix is formed all the 15 nonzero columns of length 4, taken in the lexicographic order (from 0001 to 1111).
- What is the dimension k and distance d of the code (explain your answers).
- Write out a generator matrix of the code such that the message bits are bits $1, 2, \ldots, k$.
- You are given a received vector $z = 000000 * *0000111$ where $*$ stands for erasure. Perform a decoding of z with the code. What is/are the candidate codeword(s)? Explain your answer.

(7) Consider the following Generator matrix of a code:

$$G = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

- What is the length (n) and dimension (k) of the code? How many codewords are there?
- Write down a parity-check matrix for the code.
- What is the minimum distance of this code? How many errors can this code correct? How many erasures can this code correct?
- Decode the following received word: 111000, to a valid codeword.