FLP

What about the asynchronous model?

Theorem

There is no deterministic protocol that solves Consensus in a message-passing asynchronous system in which at most one process may fail by crashing

(Fisher, Lynch, and Paterson. Impossibility of distributed consensus with one faulty process. JACM, Vol. 32, no. 2, April 1985, pp. 374-382)

The Intuition

- In an asynchronous system, a process p cannot tell whether a non-responsive process q has crashed or it is just slow
- If p waits, it might do so forever
- If p decides, it may find out later that q came to a different decision

The Model - 1

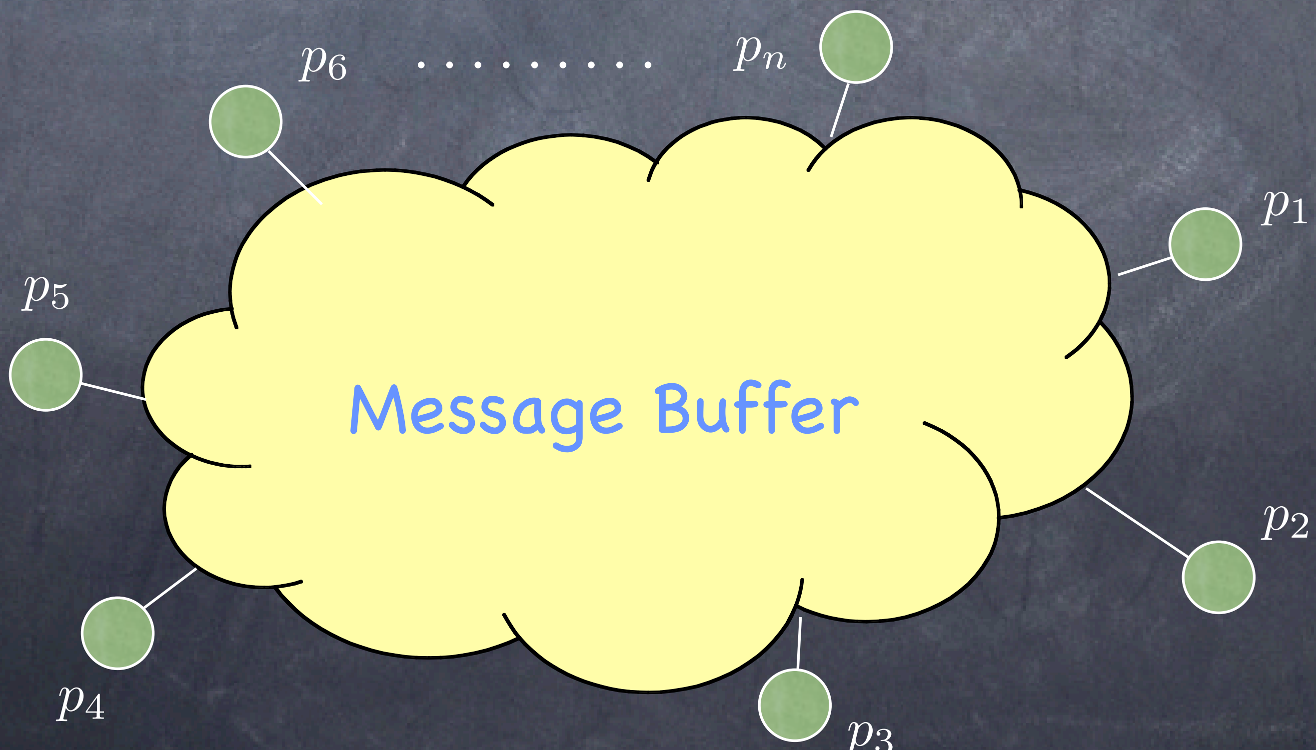
- n processes
- a message buffer

message: (p, data, q) or λ

null message

sender

receiver



The Model - 2

- An algorithm A is a sequence of steps
- Each step consists of two phases
  - Receive phase - some p removes from buffer $(x, data, p)$ or λ
  - Send phase - p changes its state; adds zero or more messages to buffer
- p can receive λ <u>even if there are messages for p in the buffer</u>

Assumptions

Liveness Assumption:

Every message sent will be eventually received if intended receiver tries infinitely often

One-time Assumption:

p sends m to q at most once

WLOG, process p_i can only propose a single bit b_i

Configurations

- A configuration C of A is a pair (s, M) where:
  - s is a function that maps each p_i to its local state
  - M is the set of messages in the buffer
- A step $e \equiv (p, m, A)$ is applicable to $C = (s, M)$ if and only if $m \in M \cup \{\lambda\}$. Note: (p, λ, A) is always applicable to C
- $C' \equiv e(C)$ is the configuration resulting from applying e to C

Schedules

- A schedule S of $\mathcal{A}$ is a finite or infinite sequence of steps of $\mathcal{A}$
- A schedule S is applicable to a configuration C if and only if either
  - □ S is the empty schedule $S_\perp$ or
  - □ $S[1]$ is applicable to C ;
  - □ $S[2]$ is applicable to $S[1](C)$; etc.
- If S is finite, $S(C)$ is the unique configuration obtained by applying S to C

Schedules and configurations

- A configuration C' is accessible from a configuration C if there exist a schedule S such that $C' = S(C)$
- C' is a configuration of $S(C)$ if $\exists S'$ prefix of S such that $S'(C) = C'$

Runs

- A run of $\mathcal{A}$ is a pair < I, S > where
  - I is an initial configuration
  - S is an infinite schedule of $\mathcal{A}$ applicable to I
- A run is partial if S is a finite schedule of $\mathcal{A}$
- A run is admissible if every process, except possibly one, takes infinitely many steps in S
- An admissible run is unacceptable if every process, except possibly one, takes infinitely many steps in S without deciding

Structure of the proof

- Show that, for any given consensus algorithm $\mathcal{A}$, there always exists an unacceptable run
- In fact, we will show an unacceptable run in which no process crashes!

Classifying Configurations

0-valent: A configuration C is 0-valent if some process has decided 0 in C, or if all configurations accessible from C are 0-valent

1-valent: A configuration C is 1-valent if some process has decided 1 in C, or if all configurations accessible from C are 1-valent

Bivalent: A configuration C is bivalent if some of the configurations accessible from it are 0-valent while others are 1-valent

Bivalent initial configurations happen

Lemma 1

There exists a bivalent initial configuration

Proof

- Suppose A solves consensus with 1 crash failure
- Let I_j be the initial configuration in which the first j b_i's are 1
- I_0 is 0-valent; I_n is 1-valent
- By contradiction, suppose no bivalent

Proof

- Suppose A solves consensus with 1 crash failure
- Let I_j be the initial configuration in which the first j b_i's are 1
- I_0 is 0-valent; I_n is 1-valent
- By contradiction, suppose no bivalent
- Let k be smallest index such that I_k is 1-valent
- Obviously, I_{k-1} is 0-valent
- Suppose p_k crashes before taking any step.
- Since A solves consensus even with one crash failure, there is a finite schedule S applicable to I_k that has no steps of p_k and such that some process decides in $S(I_k)$
- S is also applicable to I_{k-1}

CONTRADICTION

Commutativity Lemma

Lemma 2

Let S_1 and S_2 be schedules applicable to some configuration C, and suppose that the set of processes taking steps in S_1 is disjoint from the set of processes taking steps in S_2.

Then, $S_1; S_2$ and $S_2; S_1$ are both sequences applicable to C, and they lead to the same configuration.

Procrastination Lemma

Lemma 3

Let C be bivalent, and let e be a step applicable to C.

Then, there is a (possibly empty) schedule S not containing e such that $e(S(C))$ is bivalent

Proof Sketch - 1

- By contradiction, assume there is an e for which no such S exists
- Then, $e(C)$ is monovalent; WLOG assume 0-valent

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Mini Lemma:

There exists an e-free schedule S_0 such that $D = S_0(C)$ and $e(D)$ is 1-valent

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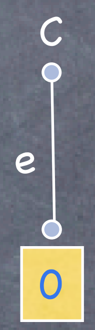

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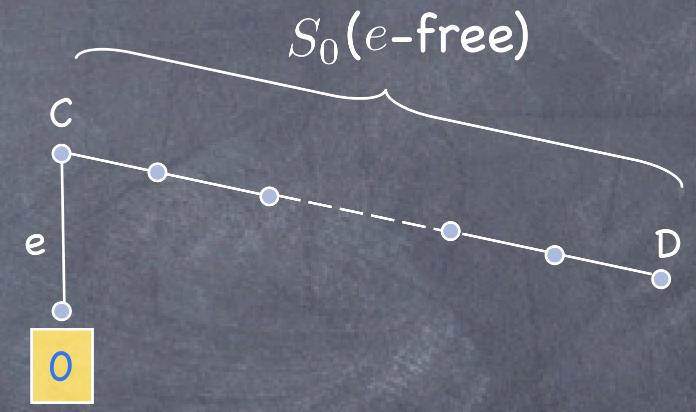

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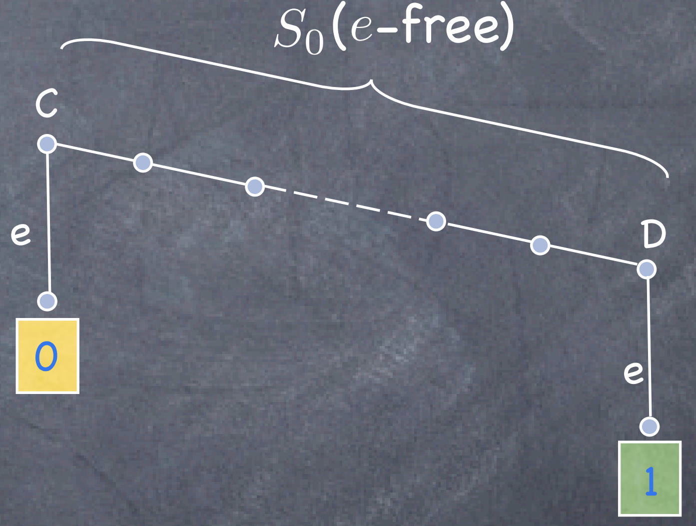

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Proof Sketch- 2

Proof of mini Lemma.

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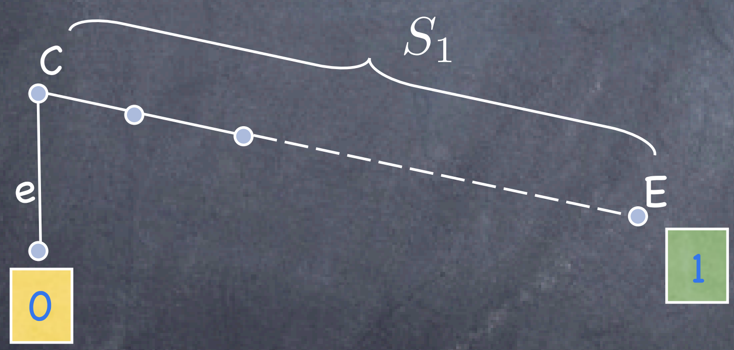

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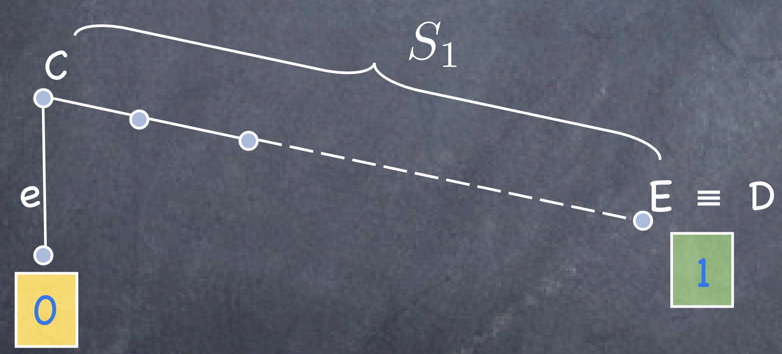

If S_1 is e-free, then D = E

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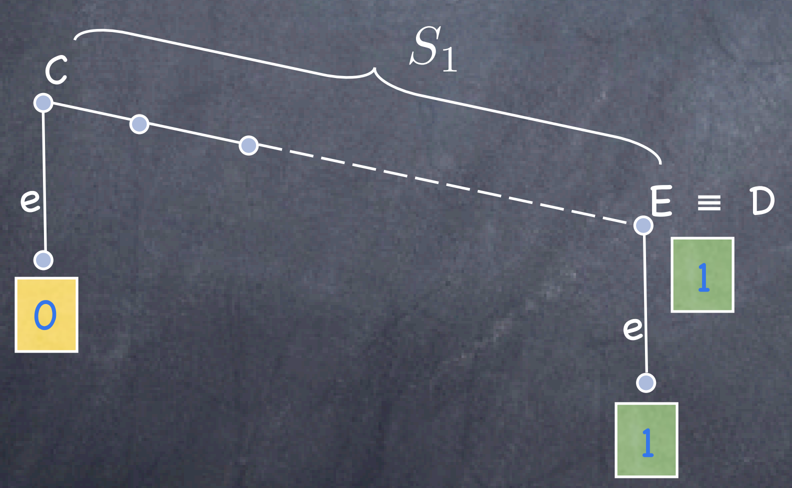

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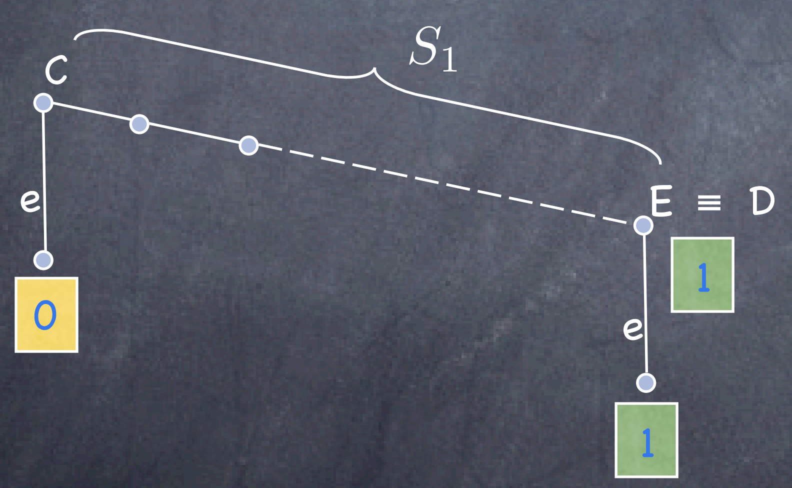

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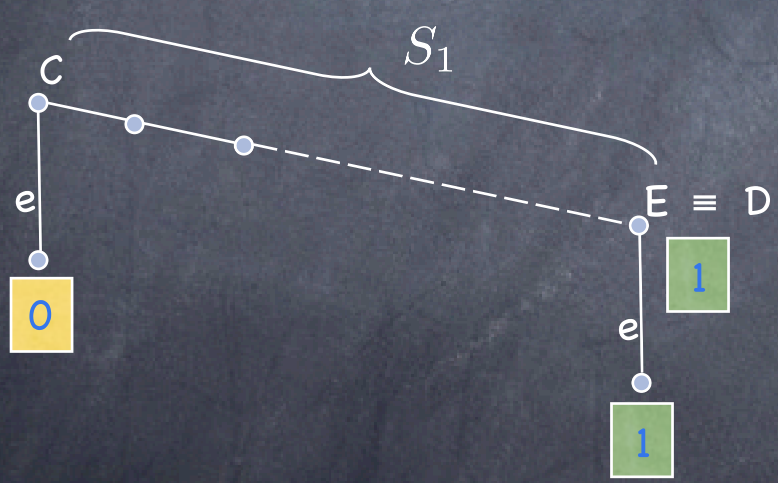

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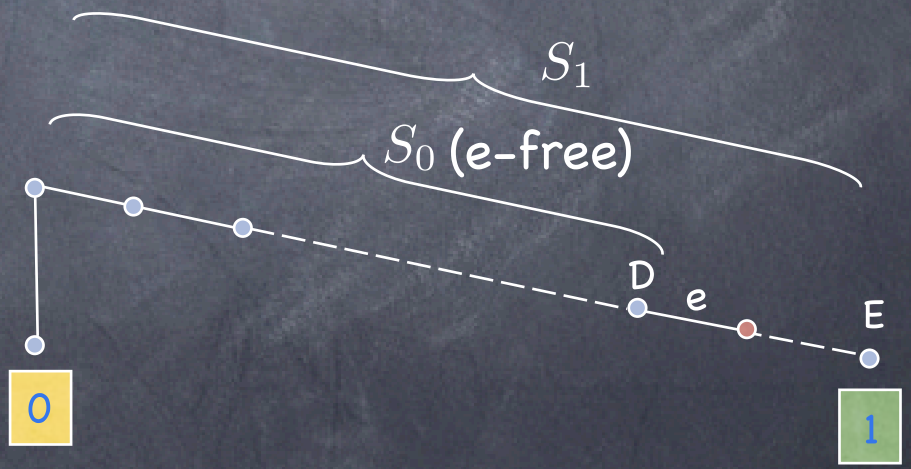
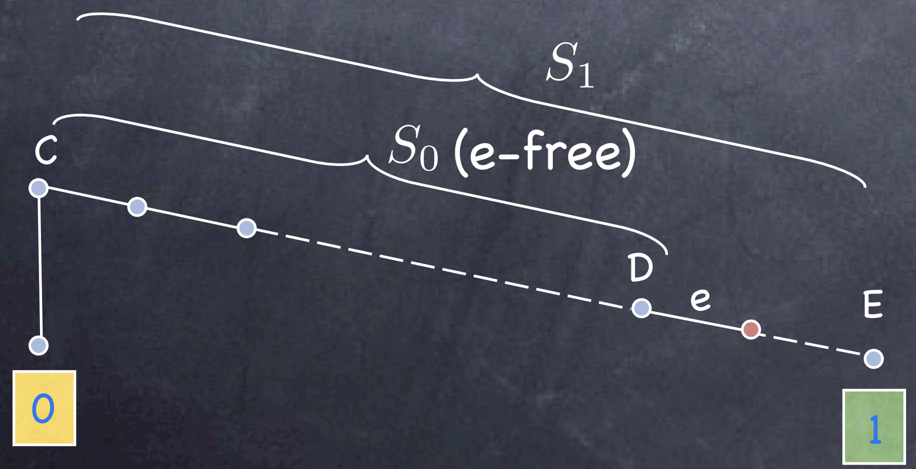


Proof Sketch - 3

- Consider configuration e(D).
- By assumption, e(D) cannot be bivalent (otherwise we would have proved the Procrastination Lemma with S = S_0)
- Since e(D) is monovalent, E is accessible from e(D), and E is 1-valent, then e(D) is 1-valent $\square$



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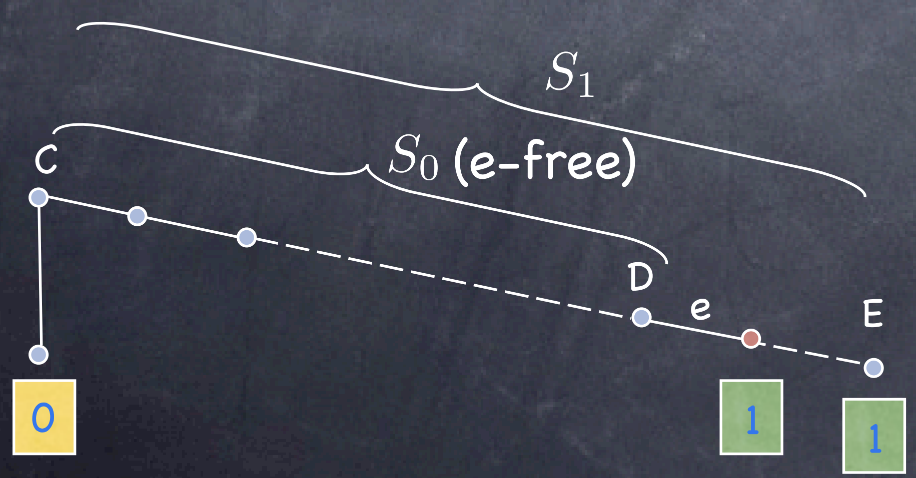

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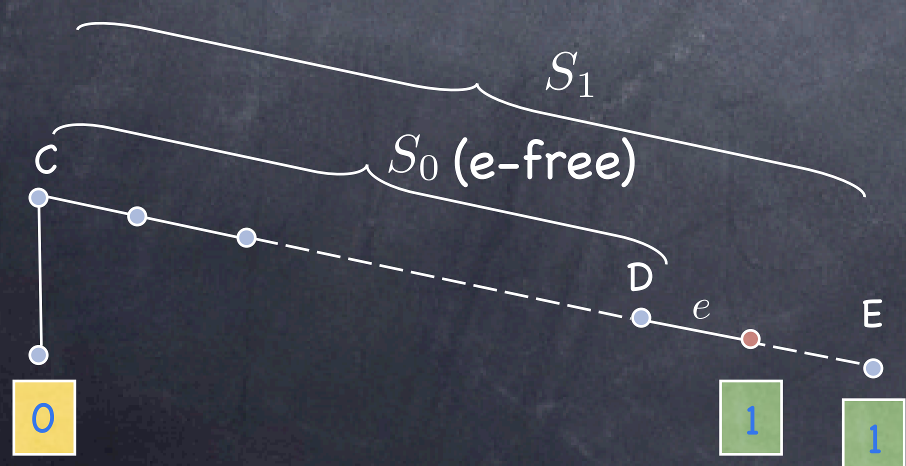
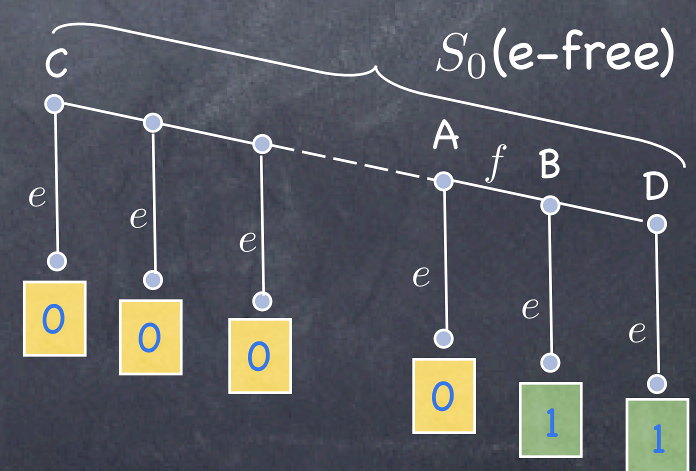


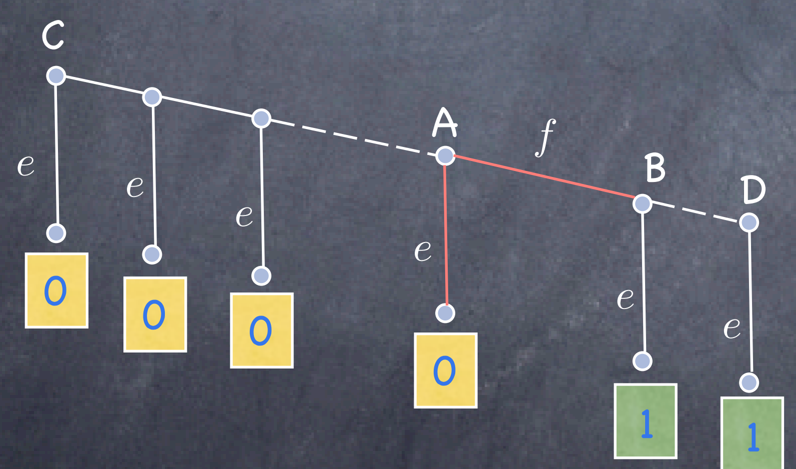
By the mini Lemma, on the "path" from C to D there must be two neighboring configurations A and B and a step f such that

- B = f(A)
- e(A) is 0-valent
- e(B) is 1-valent



Proof Sketch - 4

Consider now A and $B = f(A)$



- Claim: The same processes p must take steps e and f

Proof Sketch - 4

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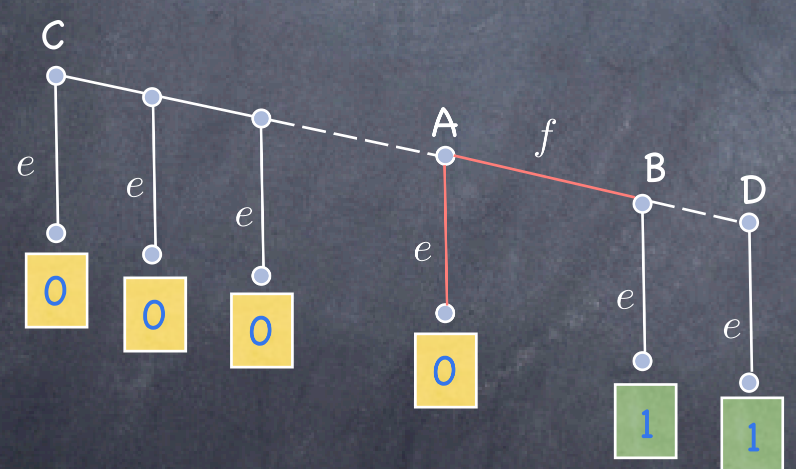

- Claim: The same processes p must take steps e and f
  - Suppose not
  - By Commutativity lemma, $e(B) = e(f(A)) = f(e(A))$

Proof Sketch - 4

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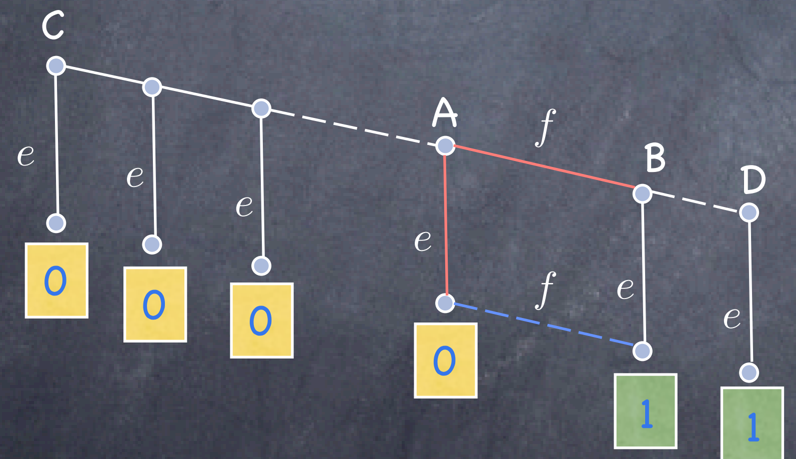

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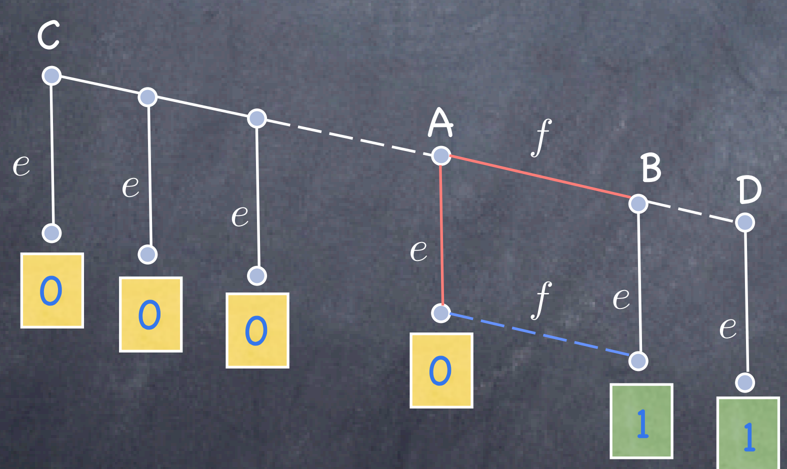
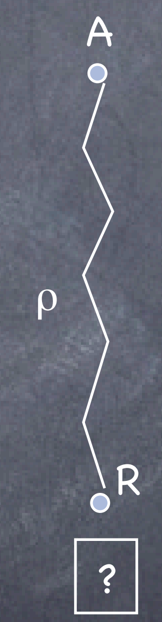


- Claim: The same processes p must take steps e and f
  - Suppose not
  - By Commutativity lemma, $e(B) = e(f(A)) = f(e(A))$
  - Impossible since $e(B)$ is 1-valent and $e(A)$ is 0-valent

Proof Sketch - 5

- Since our protocol tolerates a failure, there is a schedule ρ applicable to A such that:
  - $R = \rho(A)$
  - Some process decides in R
  - p does not take any steps in ρ



Proof Sketch - 5

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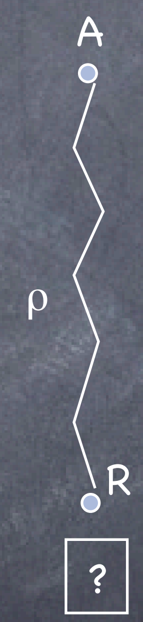
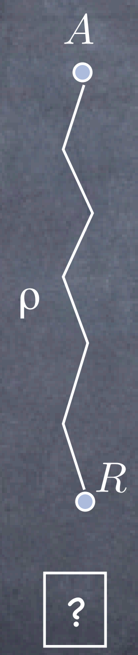


- We show that the decision value in R can be neither 0 nor 1!

Proof Sketch - 6



Cannot be 0:

- Consider $e(B) = e(f(A))$

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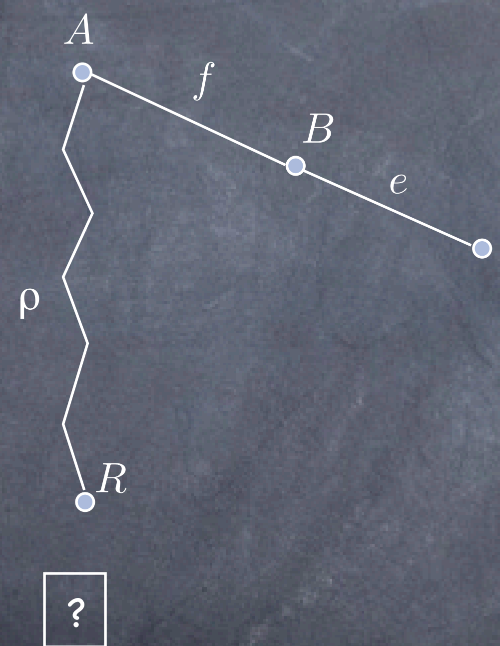

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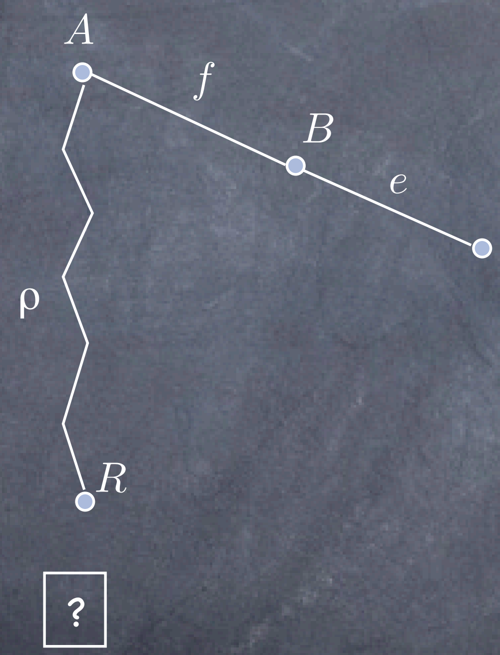

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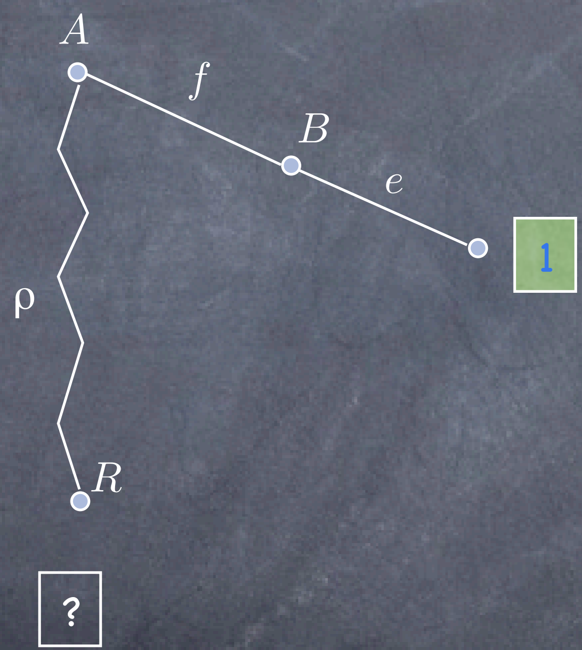

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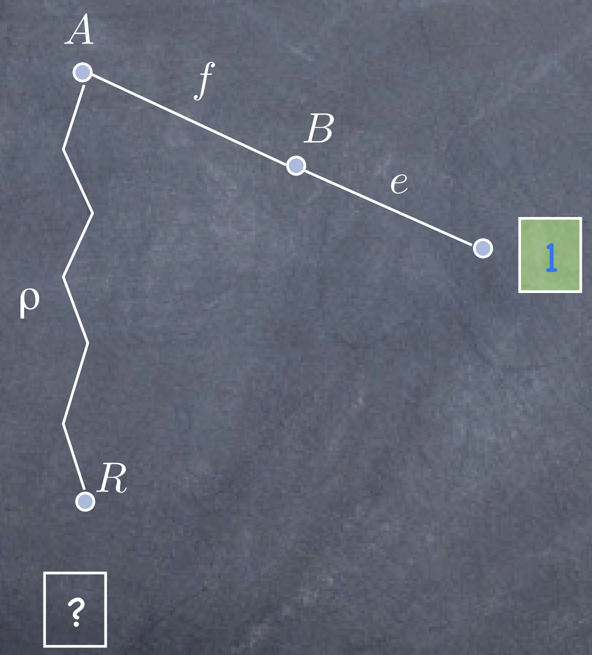

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- Consider $e(B) = e(f(A))$
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- Because it contains no steps of p , ρ is applicable to $e(B)$

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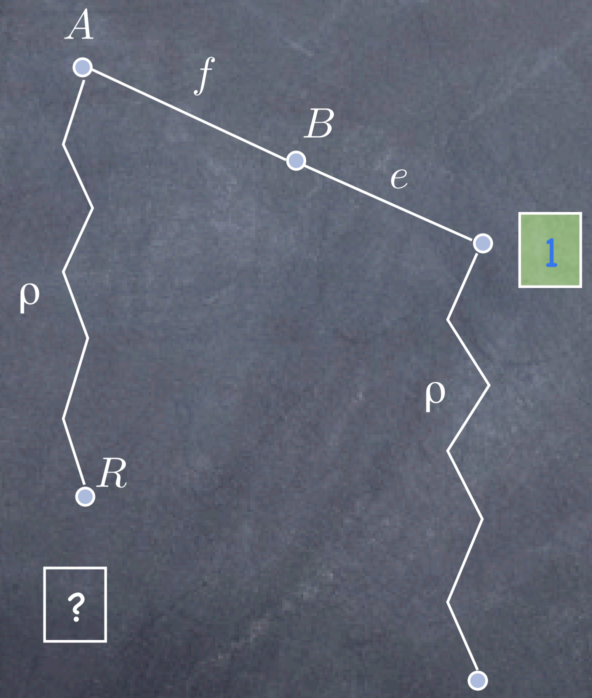

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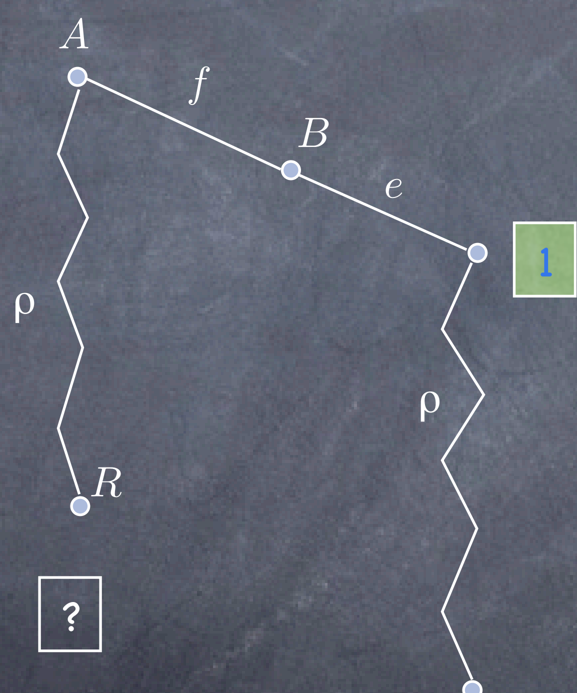

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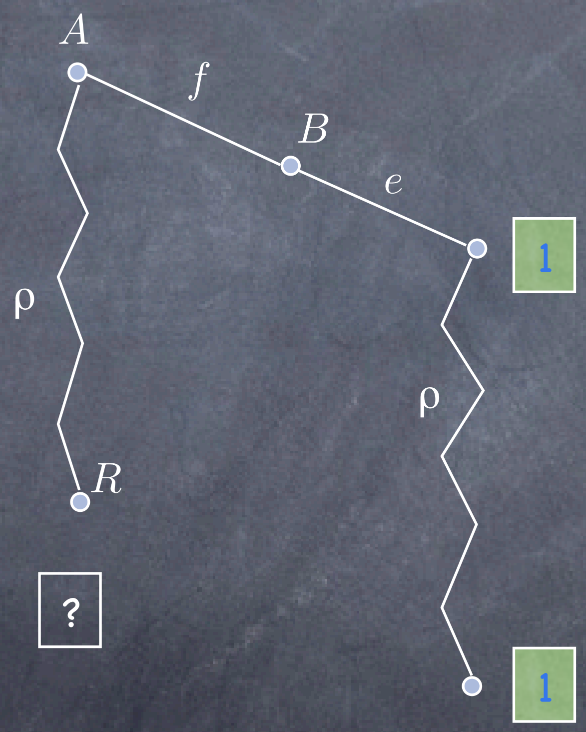

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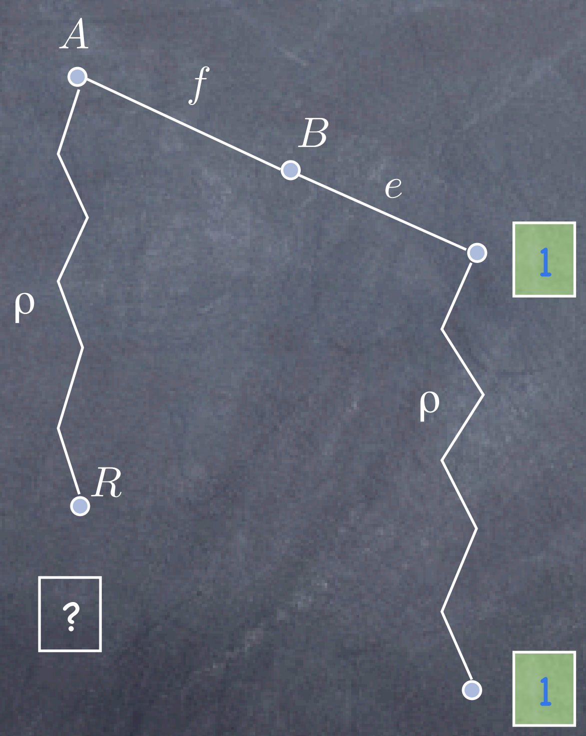

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- By Commutativity Lemma $\rho(e(f(A))) = e(f(\rho(A))) = e(f(R))$

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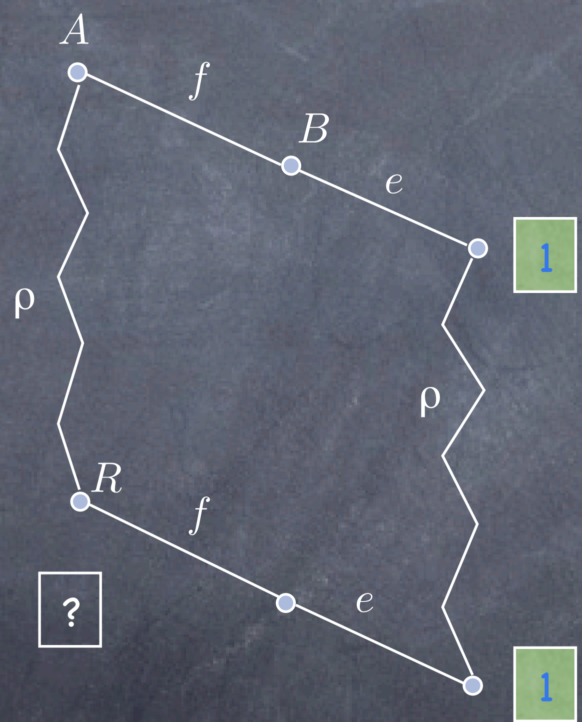

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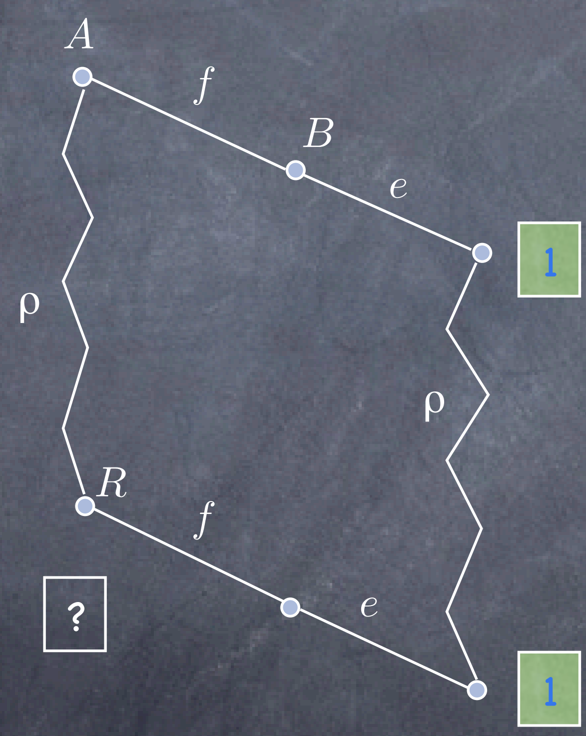

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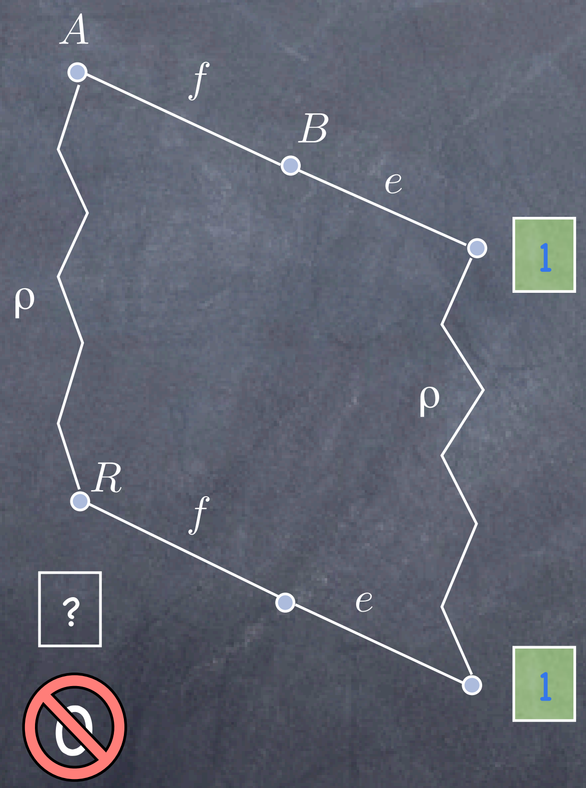
- By Commutativity Lemma $\rho(e(f(A))) = e(f(\rho(A))) = e(f(R))$
- Since $\rho(e(B))$ is accessible from R , and $\rho(e(B))$ is 1-valent, R cannot be 0-valent

Proof Sketch - 6



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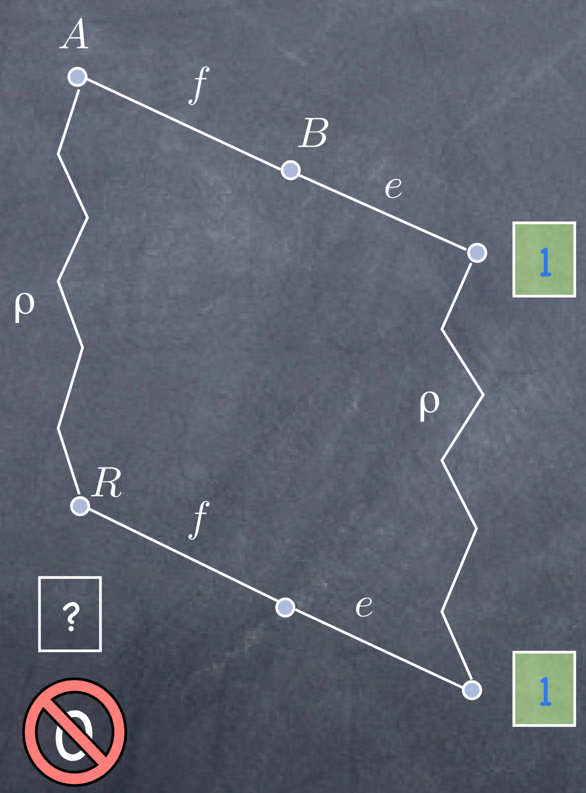
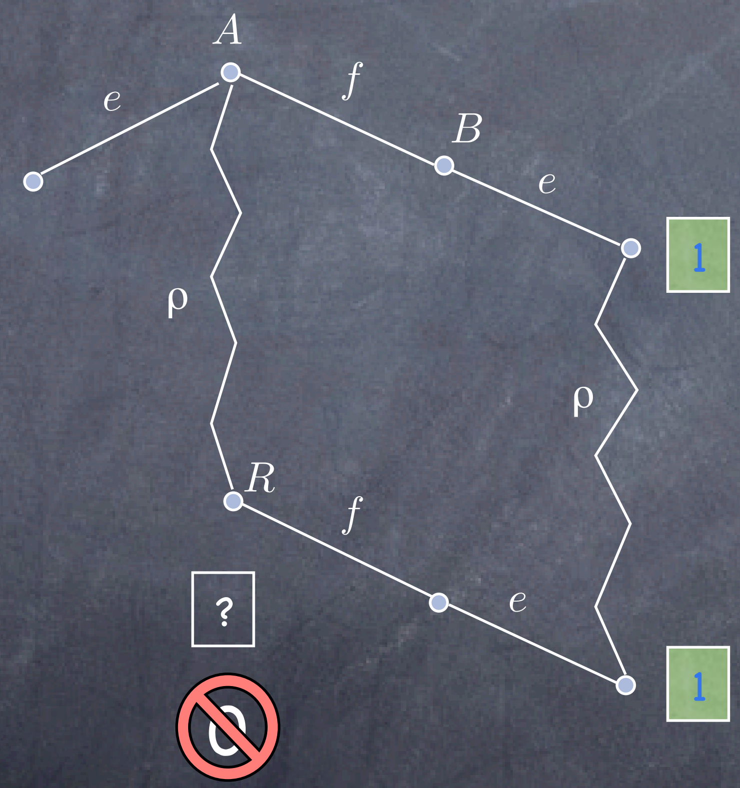


Cannot be 1:

- Consider $e(A)$

Proof Sketch - 7



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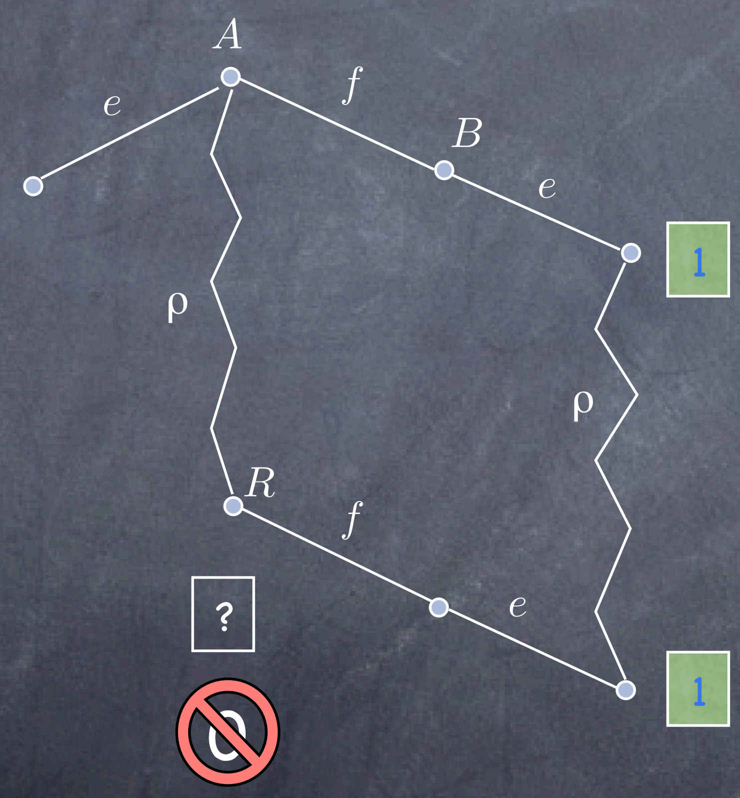
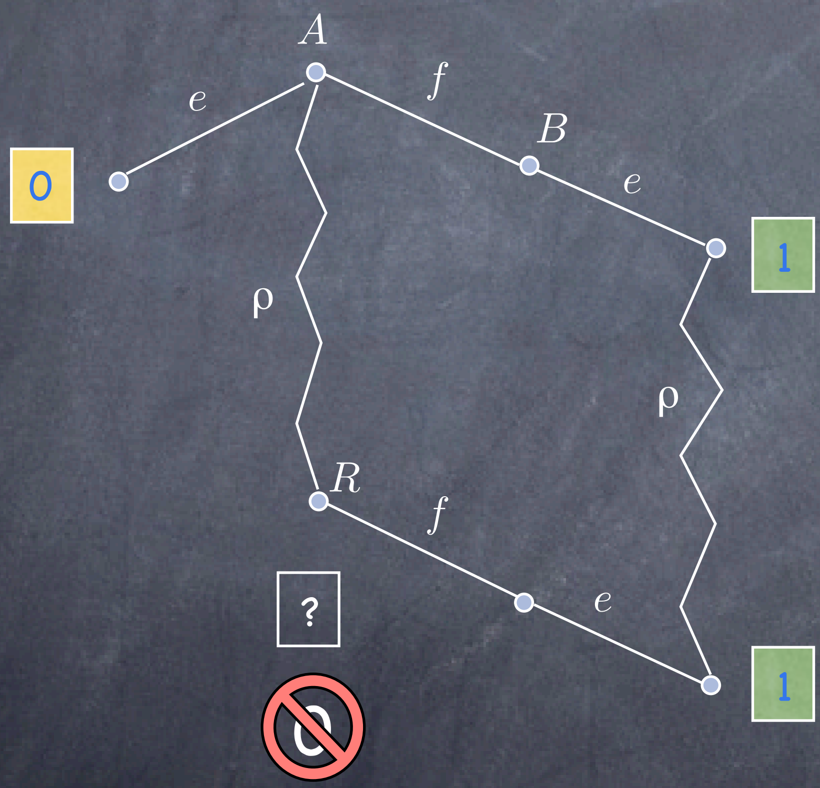


Cannot be 1:

- Consider $e(A)$
- By construction, it is 0-valent

Proof Sketch - 7



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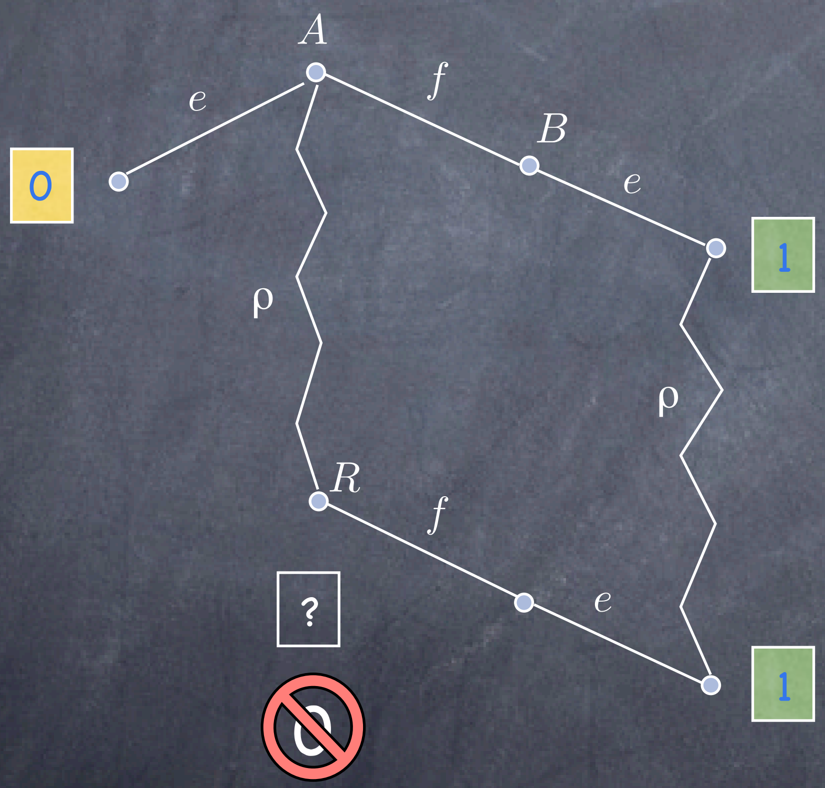

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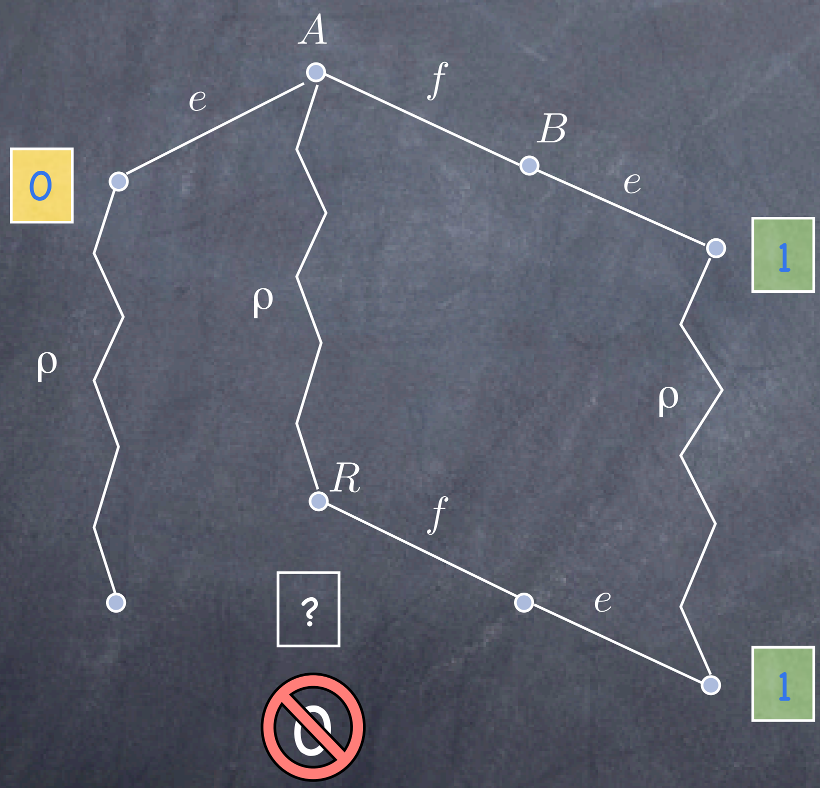

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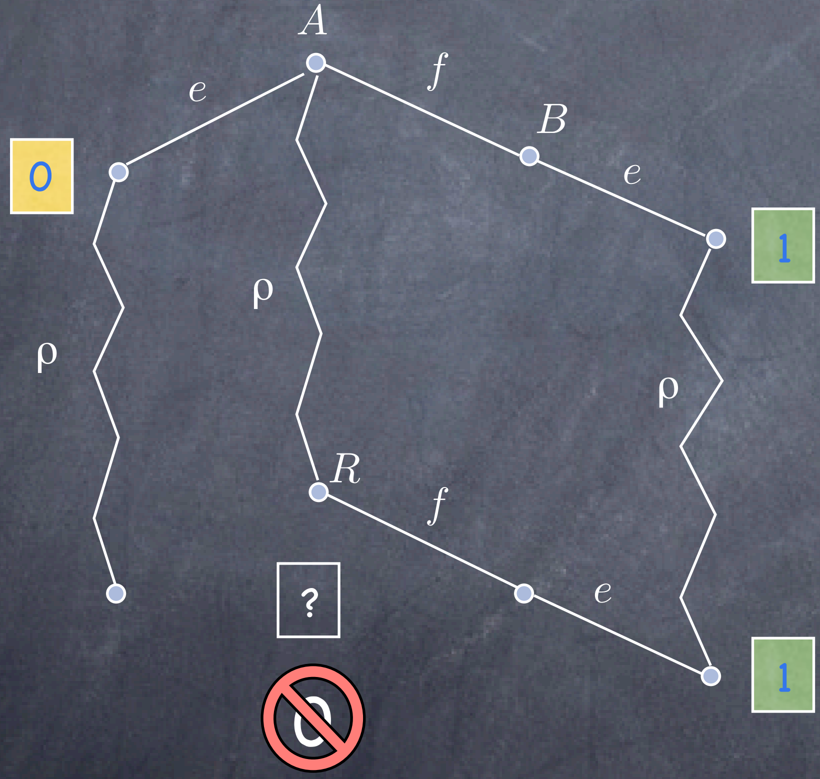
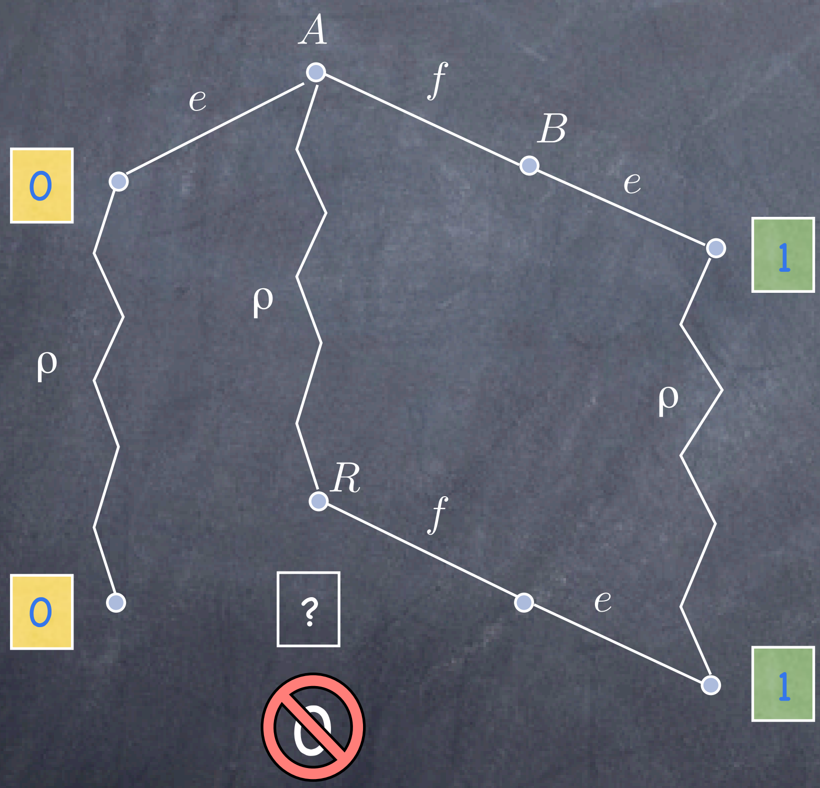


Cannot be 1:

- Consider $e(A)$
- By construction, it is 0-valent
- Because it contains no steps of p , ρ is applicable to $e(A)$
- The resulting configuration is 0-valent

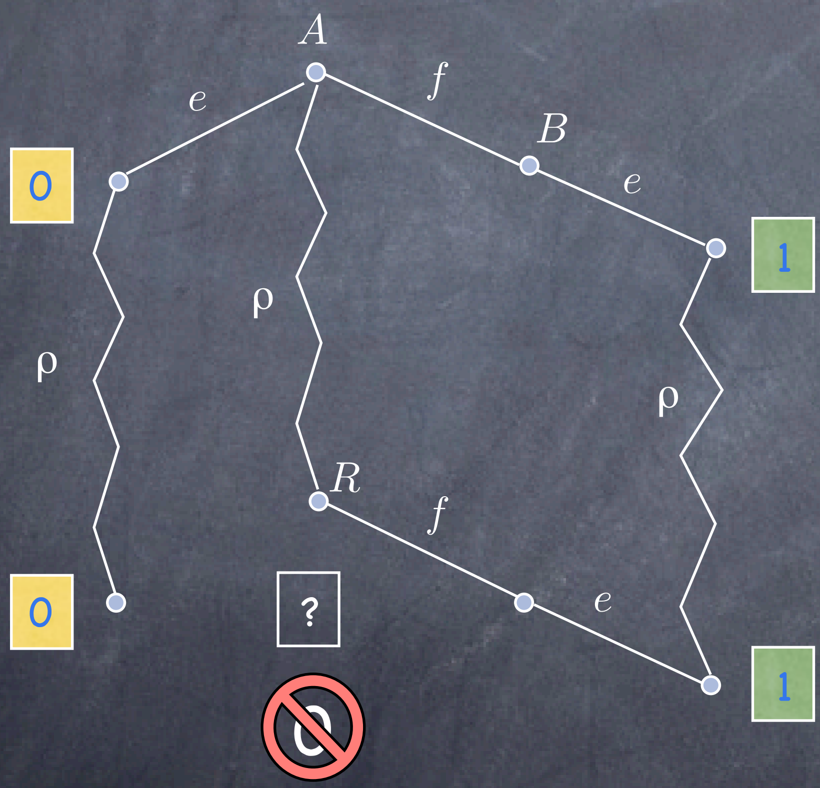
Proof Sketch - 7



Cannot be 1:

- Consider $e(A)$
- By construction, it is 0-valent
- Because it contains no steps of p , ρ is applicable to $e(A)$
- The resulting configuration is 0-valent

Proof Sketch - 7



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- By Commutativity Lemma $\rho(e(A)) = e(\rho(A)) = e(R)$

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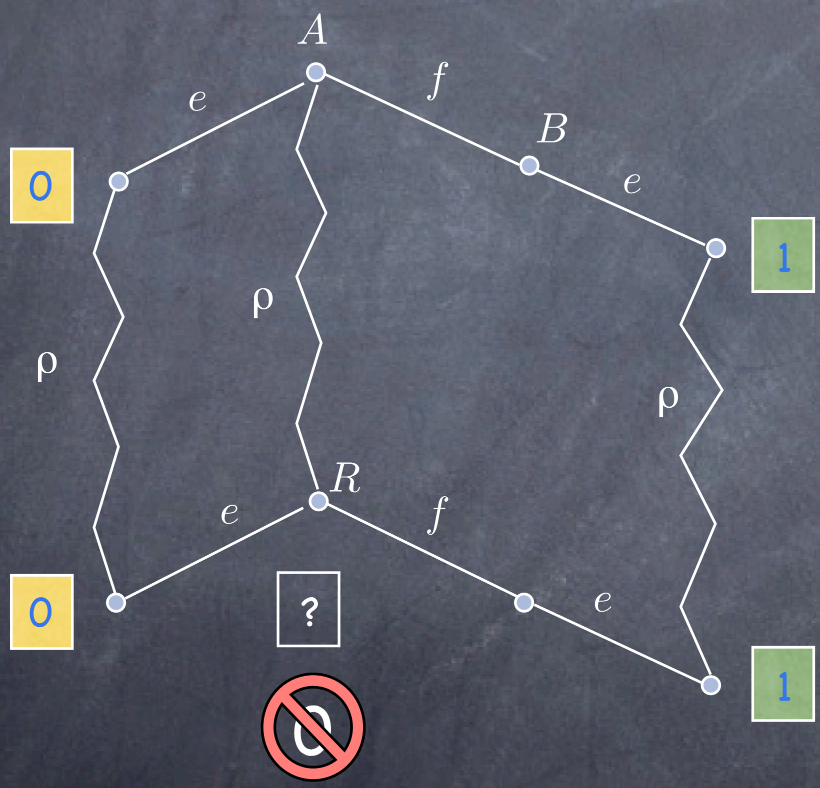

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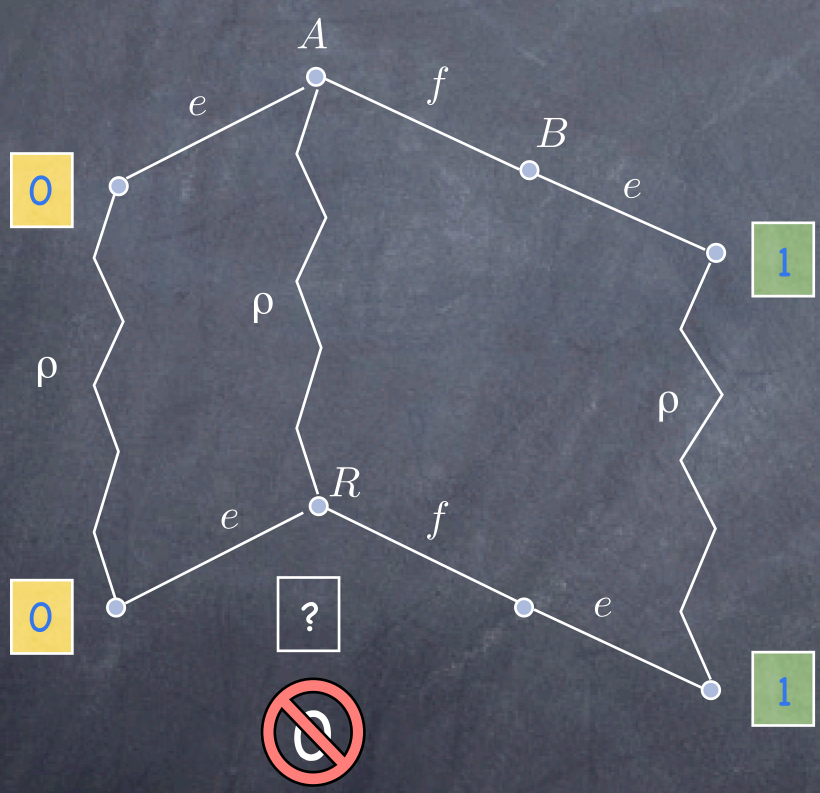

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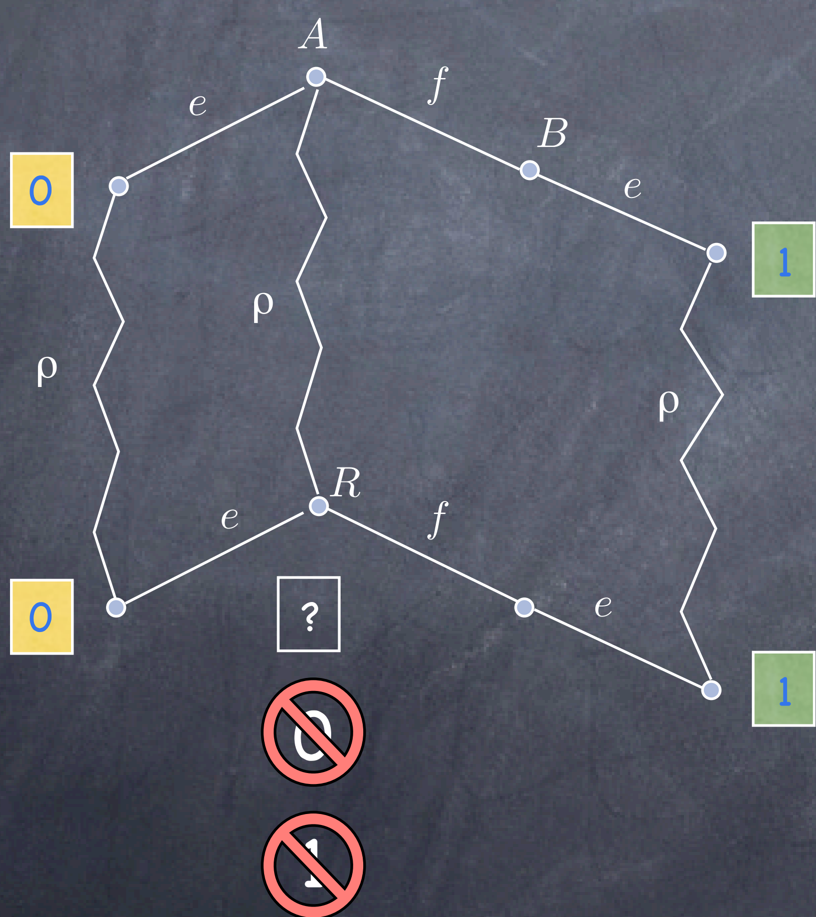
- By Commutativity Lemma $\rho(e(A)) = e(\rho(A)) = e(R)$
- Since $\rho(e(A))$ is accessible from R, and $\rho(e(A))$ is 0-valent, R cannot be 1-valent

Proof Sketch - 7



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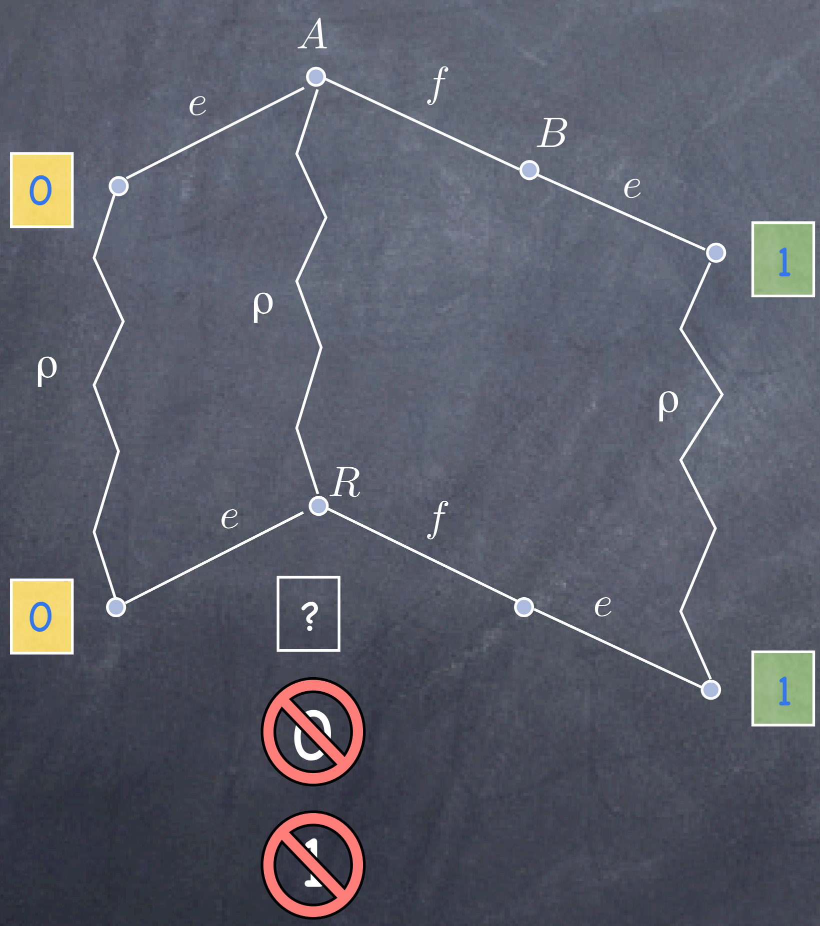

Cannot decide in R: contradiction

Cannot be 1:

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Proving the FLP Impossibility Result

Theorem

There is no deterministic protocol that solves Consensus in a message-passing asynchronous system in which at most one process may fail by crashing

- By Lemma 1, there exists an initial bivalent configuration I_{biv}
- Consider any ordering $p_{l_1}, \ldots, p_{l_n}$ of $p_1, \ldots, p_n$
- Pick any applicable step $e_1 = (p_{l_1}, m_1)$
- Apply Procrastination lemma to obtain another bivalent configuration

$$C^1_{biv} = e_1(S_1(I_{biv}))$$

- Pick a step $e_2 = (p_{l_2}, m_2)$ applicable to C^1_{biv}
- Apply Procrastination lemma to obtain another bivalent configuration
- Continue as before in a round-robin fashion. How do we choose a step?
- We have built an unacceptable run!