CMPSCI 311: Introduction to Algorithms

Lecture 16: Network Flows

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Algorithm Design Techniques

- Greedy
- Divide and Conquer
- Dynamic Programming
- Network Flows

Network Flow

- Previous topics (greedy, dynamic programming, divide and conquer etc.) were design techniques.
- Network flow relates to a specific class of problems with many applications

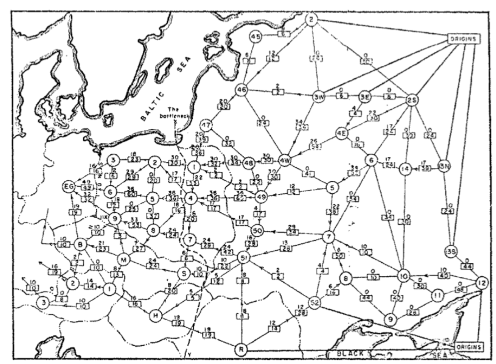
- Direct applications: commodities in networks
  - transporting food on the rail network
  - packets on the internet
  - gas through pipes
- Indirect applications:
  - Matching in graphs
  - Airline scheduling
  - Baseball elimination

Plan: design and analyze algorithms for max-flow problem, then apply to solve other problems

First, a Story About Flow and Cuts

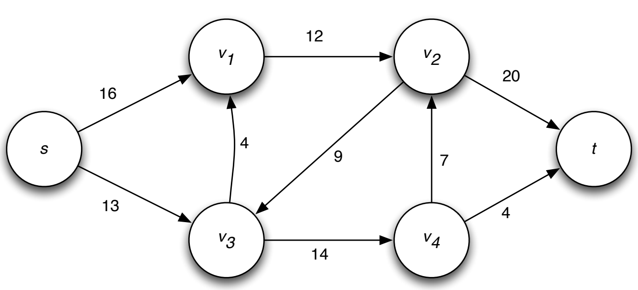
Key theme: flows in a network are intimately related to cuts

Soviet rail network in 1955

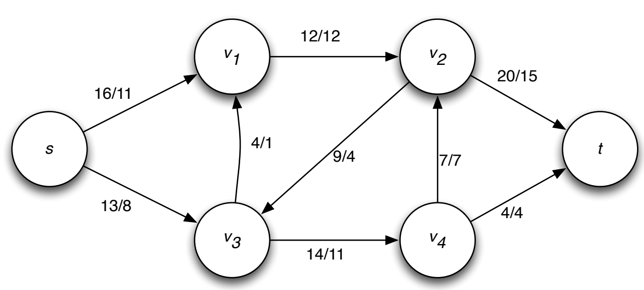


On the history of the transportation and maximum flow problems. Alexander Schrijver, Math Programming, 2002.

Capacity



Capacity/Flow



Defining Flows

- Flow network
  - Directed graph
  - Source node s and target node t
  - Edge capacities $c(e) \geq 0$
- Flow
  - Capacity Constraints: $0 \leq f(e) \leq c(e)$ on each edge
  - Conservation Constraints:

    $$f^{in}(s) = 0 \ , \quad f^{out}(t) = 0 \ , \quad \forall v \in V \setminus \{s, t\} \ f^{in}(v) = f^{out}(v)$$

    where $f^{in}(v) = \sum_{e \text{ in to } v} f(e)$ and $f^{out}(v) = \sum_{e \text{ out of } v} f(e)$
- Max flow problem: find a flow of maximum value $v(f) = f^{out}(s)$

Designing a Max-Flow Algorithm

Something that doesn't work: Repeatedly choose paths and "augment" flow on those paths until we can no longer do so

Residual Graph

Residual graph: data structure to identify opportunities to push more flow on edges with leftover capacity or undo flow on edges already carrying flow.

Original edge $e = (u, v) \in E$

- Flow $f(e)$
- Capacity $c(e)$

Forward residual edge

- $e = (u, v)$
- *residual capacity* $c(e) - f(e)$

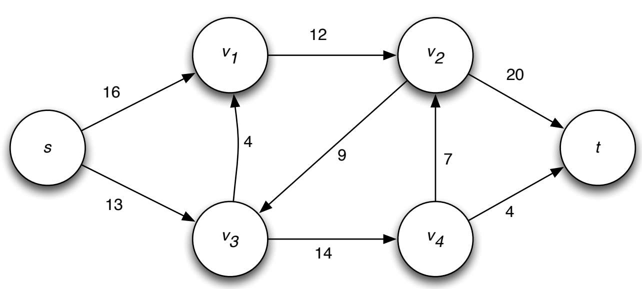
Backward residual edge

- if $f(e) > 0$, create edge $e' = (v, u)$
- *residual capacity* $f(e)$

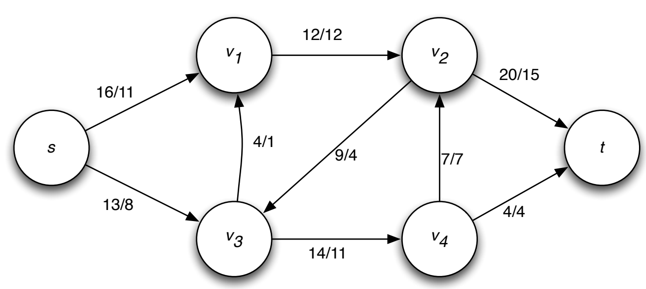
Residual Graph

Residual graph G_f with respect to flow f = graph of all forward and backward residual edges with positive residual capacity.

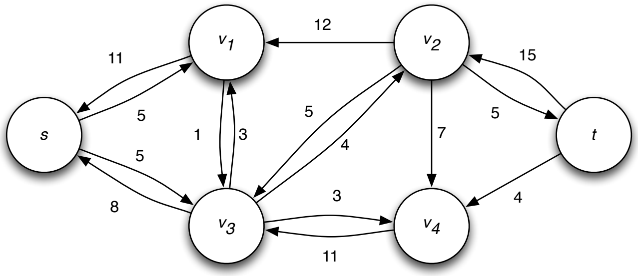
Capacity



Capacity/Flow



Residual Graph

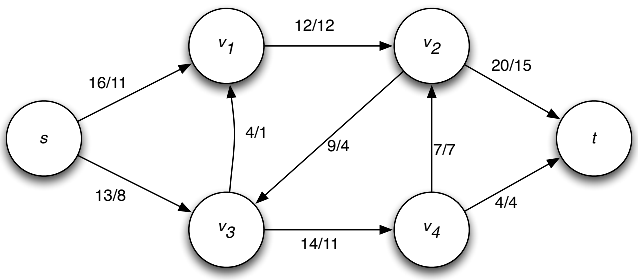


Augmenting Path

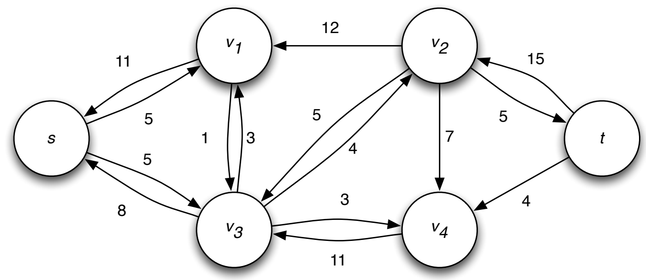
Revised Idea: use paths in the *residual* graph to augment flow

Augment(f, P)
  Let b = bottleneck(P, f) ▷ least residual capacity in P
  for edge $e = (u, v)$ in P **do**
    if e is a forward edge **then**
      $f(e) = f(e) + b$ ▷ increase flow on forward edges
    else
      $f(e) = f(e) - b$ ▷ decrease flow on backward edges
    end if
  end for

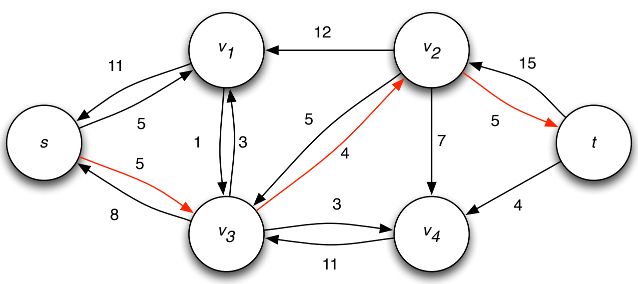
Capacity/Flow



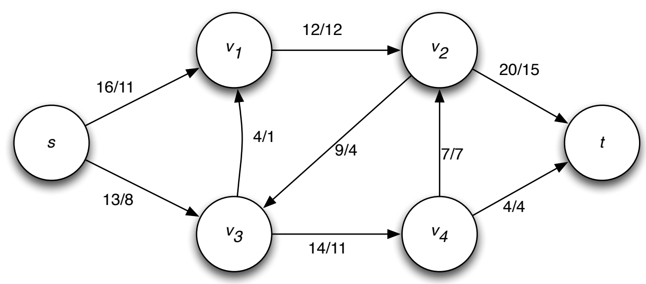
Residual



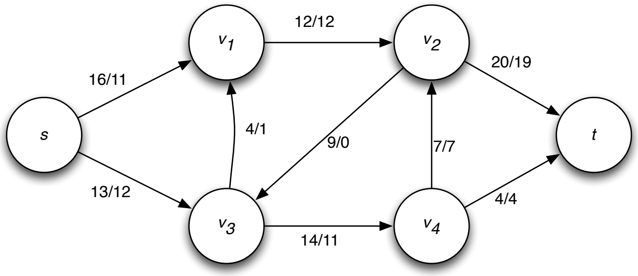
Augmenting Path



Old Flow



New Flow



Ford-Fulkerson Algorithm

Repeatedly find augmenting paths in the residual graph and use them to augment flow!

Ford-Fulkerson(G, s, t)
  ▷ Initially, no flow
  Initialize $f(e) = 0$ for all edges e
  Initialize $G_f = G$

  ▷ Augment flow as long as it is possible
  while there exists an s-t path P in G_f **do**
    f = Augment(f, P)
    update G_f
  end while
  return f

Ford-Fulkerson Analysis

- Step 1: argue that F-F returns a flow
- Step 2: analyze termination and running time
- Step 3: argue that F-F returns a maximum flow

Step 1: F-F returns a flow

Claim: If f is a flow then f' = Augment(f, P) is also a flow.

Proof idea. Verify two conditions for f' to be a flow:

- f' satisfies capacity constraints: We add at most $c(e) - f(e)$ flow along a forward edge that already has $f(e)$ flow so flow doesn't increase above $c(e)$. We add at most $f(e)$ along a backwards edge and hence flow doesn't decrease below 0.
- f' satisfies flow conservation: the extra flow into a node equals the extra flow out of a node and hence flow is still conserved.

Step 2: Termination and Running Time

Assumption: All capacities are integers. By nature of F-F, all flow values and residual capacities remain integers during the algorithm.

Running time:

- In each F-F iteration, flow increases by at least 1. Therefore, number of iterations is at most $v(f^*)$, where f^* is the final flow.
- Let C be the total capacity of edges leaving source s
- Then $v(f^*) \leq C$.
- So F-F terminates in at most C iterations

Running time per iteration? $O(m + n)$ to find an augmenting path

Step 3: F-F returns a maximum flow

We will prove this by establishing a deep connection between flows and cuts in graphs: the max-flow min-cut theorem.

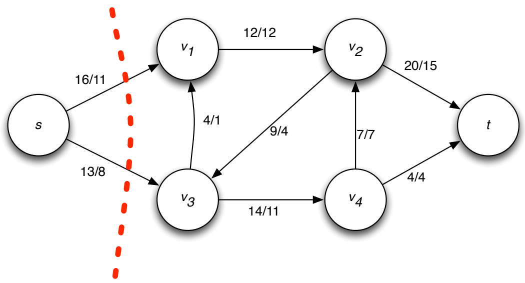
- An s-t cut (A, B) is a partition of the nodes into sets A and B where $s \in A$, $t \in B$
- Capacity of cut (A, B) equals

$$c(A, B) = \sum_{e \text{ from } A \text{ to } B} c(e)$$

- Flow across a cut (A, B) equals

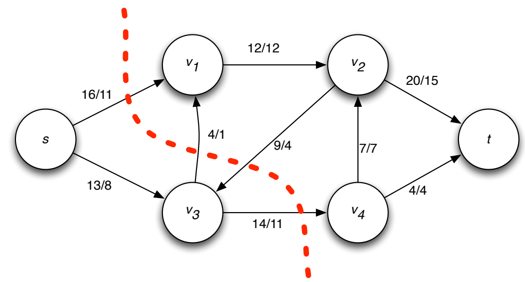
$$f(A, B) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$$

Example of Cut



Capacity is 29 and flow across cut is 19.

Another Example of Cut



Capacity is 34 and flow across cut is 19.

Flow Value Lemma

First relationship between cuts and flows

Lemma: let f be any flow and (A, B) be any s-t cut. Then

$$v(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$$

Basic idea of proof is to use conservation of flow: all the flow out of s must leave A eventually.

Corollary: Cuts and Flows

Really important corollary of flow-value lemma

Corollary: Let f be **any** s-t flow and let (A, B) be **any** s-t cut. Then $v(f) \le c(A, B)$.

Proof:

$$\begin{aligned} v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) \\ &\le \sum_{e \text{ out of } A} f(e) \\ &\le \sum_{e \text{ out of } A} c(e) \\ &= c(A, B) \end{aligned}$$

Implies that if there's a flow f^* and cut (A^*, B^*) with $v(f^*) = c(A^*, B^*)$, then f^* is a max flow and (A^*, B^*) is a min cut.

F-F returns a maximum flow

Theorem: The s-t flow f^* returned by F-F is a maximum flow.

- Since f^* is the final flow there are no residual paths in G_{f^*}.
- Let (A^*, B^*) be the s-t cut where A^* consists of all nodes reachable from s in the residual graph.
- Then $v(f) = f(A^*, B^*) = \sum_{e \text{ out of} A^*} f(e) - \sum_{e \text{ into} A^*} f(e)$.
- Any edge out of A^* must have $f(e) = c(e)$ otherwise there would be more nodes than just A^* that reachable from s.
- Any edge into A^* must have $f(e) = 0$ otherwise there would be more nodes than just A^* that reachable from s.
- Therefore $v(f) = f(A^*, B^*) = \sum_{e \text{ out of} A^*} f(e) - \sum_{e \text{ into} A^*} f(e) = \sum_{e \text{ out of} A^*} c(e) = c(A^*, B^*)$.