CMPSCI 311: Introduction to Algorithms

Lecture 10: More Divide and Conquer

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Last Compiled: February 28, 2018

Integer Multiplication

Motivation: multiply two 30-digit integers?

```
  153819617987625488624070712657
x 925421863832406144537293648227
  ------------------------------
```

- Multiply two 300-digit integers?
- Cannot do this in Java with built-in data types
- 64-bit unsigned integer can only represent integers up to ~20 digits ($2^{64} \approx 10^{20}$)

Warm-Up: Addition

Input: two n-digit binary integers x and y
Goal: compute $x + y$

Let's do everything in base-10 instead of binary to make examples more familiar.

Grade-school algorithm:

```
    1854
  + 3242
 -------
    5096
```

Running time? $\Theta(n)$

Integer Multiplication Problem

Input: two n-digit base-10 integers x and y
Goal: compute xy

Can anyone think of an algorithm?

Grade-School Algorithm (Long Multiplication)

Example: $n = 3$

```
    287
  x 132
 ------
    574
   861
  287
--------
  37884
```

$$287 \times 132 = (2 \times 287) + 10 \cdot (3 \times 287) + 100 \cdot (1 \times 287)$$

Running time? $\Theta(n^2)$
But xy has at most $2n$ digits. Can we do better?

Divide and Conquer

Idea: split x and y in half (assume n is a power of 2)

$$x = \underbrace{3380}_{x_1}\underbrace{2367}_{x_0}$$
$$y = \underbrace{4508}_{y_1}\underbrace{1854}_{y_0}$$

Then use distributive law

$$\begin{aligned} xy &= (10^{n/2}x_1 + x_0) \times (10^{n/2}y_1 + y_0) \\ &= 10^n x_1y_1 + 10^{n/2}(x_1y_0 + x_0y_1) + x_0y_0 \end{aligned}$$

Have reduced the problem to multiplications of $n/2$-digit integers and additions of n-digit numbers

Divide and Conquer: First Try

Recursive algorithm:

$$xy = 10^n x_1 y_1 + 10^{n/2}(x_1 y_0 + x_0 y_1) + x_0 y_0$$

Running time? Four multiplications of $n/2$ digit numbers plus three additions of at most $2n$-digit numbers

$$\begin{aligned} T(n) &\leq 4T\left(\frac{n}{2}\right) + cn \\ &= O(n^{\log_2 4}) \\ &= O(n^2) \end{aligned}$$

We did not beat the grade-school algorithm. :(

Better Divide and Conquer

Same starting point:

$$xy = 10^n x_1 y_1 + 10^{n/2}(x_1 y_0 + x_0 y_1) + x_0 y_0$$

Trick: use three multiplications to compute the following:

$$\begin{aligned} A &= (x_1 + x_0)(y_1 + y_0) = x_1 y_1 + x_1 y_0 + x_0 y_1 + x_0 y_0 \\ B &= x_1 y_1 \\ C &= x_0 y_0 \end{aligned}$$

Then

$$xy = 10^n B + 10^{n/2}(A - B - C) + C$$

Total: three multiplications of $n/2$-digit integers, six additions

Better Divide and Conquer

Total: three multiplications of $n/2$-digit integers, six additions of at most $2n$-digit integers

$$\begin{aligned} T(n) &\leq 3T\left(\frac{n}{2}\right) + cn \\ &= O(n^{\log_2 3}) \\ &\approx O(n^{1.59}) \end{aligned}$$

We beat long multiplication!

Idea can be generalized to be even faster (split x and y into k parts instead of two)

Finding Minimum Distance between Points on a Plane

- **Problem:** Given n distinct points $p_1, \ldots, p_n \in \mathbb{R}^2$, find

  minimum distance between any two points $= \min_{i \neq j} d(p_i, p_j)$

$$d(p, q) = \sqrt{(p[1] - q[1])^2 + (p[2] - q[2])^2}$$

How long does naive algorithm take? $O(n^2)$

We'll do it in $O(n \log n)$ steps.

Minimum Distance Algorithm

- Divide points P with a vertical line into P_L and P_R where $|P_L| = |P_R| = n/2$
- Recursively find minimum distance within P_L and P_R:

$$\delta_L = \min_{p,q \in P_L : p \neq q} d(p, q) \quad \delta_R = \min_{p,q \in P_R : p \neq q} d(p, q)$$

- Compute $\delta_M = \min_{p \in P_L, q \in P_R} d(p, q)$ and return

$$\min(\delta_L, \delta_R, \delta_M)$$

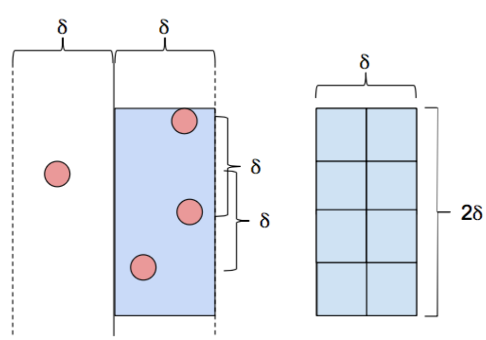
- If Step 3 takes $\Omega(n^2)$ time, we get

$$T(n) \leq 2T(n/2) + \Omega(n^2) \implies T(n) = \Omega(n^2)$$

- If we can do Step 3 in $\Theta(n)$ time, we get $T(n) = O(n \log n)$.

Making Step 3 Efficient

- Need to find $\min(\delta_L, \delta_R, \delta_M)$ where $\delta_M = \min_{p \in P_L, q \in P_R} d(p, q)$
- Suppose that the dividing line is $x = m$ and $\delta = \min(\delta_L, \delta_R)$



Making Step 3 Efficient

- Need to find $\min(\delta_L, \delta_R, \delta_M)$ where $\delta_M = \min_{p \in P_L, q \in P_R} d(p, q)$
- Suppose that the dividing line is $x = m$ and $\delta = \min(\delta_L, \delta_R)$
- Once we know δ, only need $O(n)$ comparisons to find $\min(\delta, \delta_M)$
  - Only compare $(p_1, p_2) \in P_L, (q_1, q_2) \in P_R$ if

$$m - \delta < p_1 \leq q_1 < m + \delta \quad \text{and} \quad |p_2 - q_2| < \delta$$

  - Each point $p \in P_L$ only gets compared with $O(1)$ points in P_R
- Need to identify the relevant comparisons in $O(n)$ time
  - Make two copies of points sorted by each coordinate
  - Ensure both lists are passed to each recursion sorted
  - Given sorted lists, it's easy to find the relevant points

Merge step pseudocode

- Assume P_L, P_R sorted in increasing by second coordinate.
- Assume they only contain the points within δ of the boundary.

```
Lw = [], Rw =[], δ_M = ∞
while P_L.next(),P_R.next() ≠ None do
    if P_L.next().y < P_R.next().y then
        Append next = P_L.pop() to Lwindow;
        Remove points in Lw, Rw with y-distance > δ from next
        Compare distances between Lw, Rw, update δ_M.
    else                                  ▷ Same thing but for P_R.next()
    end if
end while
```

- **Fact.** Lw, Rw always of $O(1)$ size!
- **Runtime.** $O(n \log n)$.