CMPSCI 311: Introduction to Algorithms	Spring 2018

Homework 6

Released 4/18/2018 — Due 11:59pm 5/1/2018 in Gradescope

Instructions. You may work in groups, but you must individually write your solutions yourself. List your collaborators on your submission.

If you are asked to design an algorithm as part of a homework problem, please provide: (a) the pseudocode for the algorithm, (b) an explanation of the intuition for the algorithm, (c) a proof of correctness, (d) the running time of your algorithm, and (e) justification for your running time analysis.

There are 4 questions, each worth 25 points total.

Submission instructions. This assignment is due by 11:59pm on 5/1/2018 in Gradescope. Please submit a pdf file. You may submit a scanned handwritten document, but a typed submission is preferred.

1. **Hitting Sets (K&T Ch8.Ex5).** Given a set $A = \{a_1, \ldots, a_n\}$ and a collection $B_1, \ldots, B_m \subset A$ we say that $H \subset A$ is a *hitting set* for the collection if H contains at least one element from each B_i, that is $|H \cap B_i| > 0$.

   The Hitting-Set problem is the following: Given a set $A = \{a_1, \ldots, a_n\}$, subsets $B_1, \ldots, B_m \subset A$, and a number k, is there a hitting set $H \subset A$ of size at most k. Prove that Hitting-Set is NP-Complete.

2. **Path Selection (K&T Ch8.Ex9).** Consider the following problem. You are managing a communication network, modeled by a directed graph $G = (V, E)$. There are c users who are interested in making use of this network. User i (for each $i = 1, 2, ..., c$) issues a request to reserve a specific path P_i in G on which to transmit data.

   You are interested in accepting as many of these path requests as possible, subject to the following restriction: if you accept both P_i and P_j, then P_i and P_j cannot share any nodes.

   Thus, the Path Selection problem asks: Given a directed graph $G = (V, E)$, a set of requests $P_1, P_2, ..., P_c$ (each of which must be a path in G) and a number k, is it possible to select at least k of the paths so that no two of the selected paths share any nodes? Prove that Path Selection is NP-complete.

3. **Combinatorial Auctions (K&T Ch8.Ex13).** A *combinatorial auction* is a particular mechanism developed by economists for selling a collection of items to a collection of potential buyers. (The Federal Communications Commission has studied this type of auction for assigning stations on the radio spectrum to broadcasting companies.)

   Heres a simple type of combinatorial auction. There are n items for sale, labeled $I_1, ..., I_n$. Each item is indivisible and can only be sold to one person. Now, m different people place bids: The ith bid specifies a subset S_i of the items, and an offering price x_i that the bidder is willing to pay for the items in the set S_i, as a single unit. (Well represent this bid as the pair (S_i, x_i).)

   An auctioneer now looks at the set of all m bids; she chooses to accept some of these bids and to reject the others. Each person whose bid i is accepted gets to take all the items in the corresponding set S_i. Thus the rule is that no two accepted bids can specify sets that contain a common item, since this would involve giving the same item to two different people.

   The auctioneer collects the sum of the offering prices of all accepted bids. (Note that this is a one-shot auction; there is no opportunity to place further bids.) The auctioneers goal is to collect as much money as possible.

   Thus, the problem of Winner Determination for Combinatorial Auctions asks: Given items $I_1, ..., I_n$, bids $(S_1, x_1), ..., (S_m, x_m)$, and a bound B, is there a collection of bids that the auctioneer can accept so as to collect an amount of money that is at least B?

Prove that the problem of Winner Determination for Combinatorial Auctions is NP-complete.

Example. Suppose an auctioneer decides to use this method to sell some excess computer equipment. There are four items labeled `PC`, `monitor`, `printer`, and `scanner`; and three people place bids. Define

$$S_1 = [\texttt{PC,monitor}], S_2 = [\texttt{PC,printer}], S_3 = [\texttt{monitor,printer,scanner}]$$

and

$$x_1 = x_2 = x_3 = 1$$

.

The bids are $(S_1, x_1), (S_2, x_2), (S_3, x_3)$, and the bound B is equal to 2.

Then the answer to this instance is no: The auctioneer can accept at most one of the bids (since any two bids have a desired item in common), and this results in a total monetary value of only 1.

4. **3-Coloring (K&T Ch13.Ex1).** Consider the optimization version of 3-Coloring: Given a graph $G = (V, E)$, color each node with one of three colors to maximize the number of edges whose incident vertices have different colors. We say that an edge (u, v) is satisfied if the colors assigned to u and v are different.

   Suppose that $c^\star$ is the maximum number of satisfied edges. Give a polynomial time algorithm that produces a 3-coloring that satisfies $\frac{2}{3}c^\star$ edges. If your algorithm is randomized, the expected number of satisfied edges should be at least $\frac{2}{3}c^\star$.