Foundations of Applied Cryptography

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Based on http://cseweb.ucsd.edu/~mihir/cse207/

What is a “good” blockcipher?

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- Key recovery is hard.

What is a “good” blockcipher?

We want to define a notion of a “good” blockcipher, where “good” means natural uses of the blockcipher are secure.

One idea is to list requirements:

- Key recovery is hard.
- Message recovery is hard.

not very convincing.

Analogy to Intelligence

What if we want to define the notion of "intelligent" for a computer program?

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- It can be happy.
- It can multiply numbers

Analogy to Intelligence

What if we want to define the notion of "intelligent" for a computer program?

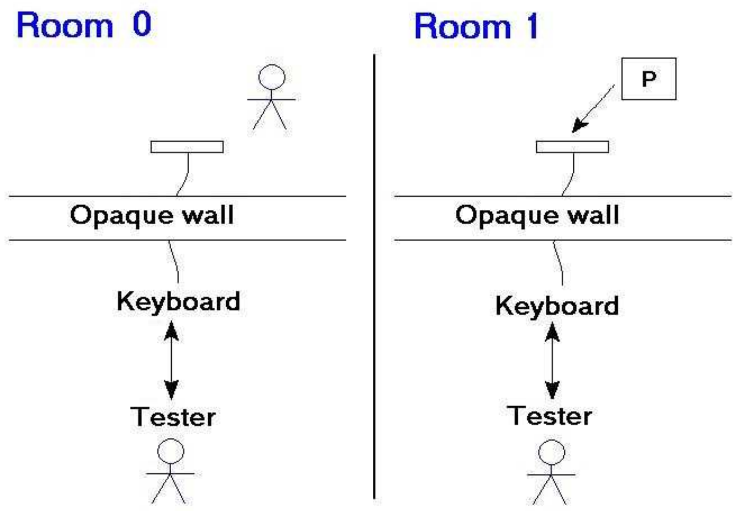
Again, one idea is to list requirements:

- It can be happy.
- It can multiply numbers
- … but only small numbers.

Turing’s Answer

A program is “intelligent” if its input/output behavior is indistinguishable from that of a human.

The Turing Test



Game:

- Put tester in room 0 and let it interact with object behind wall
- Put tester in rooom 1 and let it interact with object behind wall
- Now ask tester: which room was which?

The measure of "intelligence" of P is the extent to which the tester fails.

The Analogy

Notion	Real object	Ideal object
Intelligence	Program	Human
PRF	Block cipher	?

random function

Random Functions

lazy sampling

Game Rand_R // here R is a set

procedure Fn(x)

if $\mathsf{T}[x] = \perp$ then $\mathsf{T}[x] \xleftarrow{\$} R$

return $\mathsf{T}[x]$

Adversary A

- Make queries to **Fn**
- Eventually halts with some output

We denote by

$$\Pr\left[\mathrm{Rand}_R^A \Rightarrow d\right]$$

the probability that A outputs d

Random Functions

T initialized to ⊥ (empty)

Game $\text{Rand}_{\{0,1\}^3}$

procedure Fn(x)

if $T[x] = \perp$ then $T[x] \xleftarrow{\$} \{0,1\}^3$

return $T[x]$

adversary A

$y \leftarrow \textbf{Fn}(01)$

return $(y = 000)$

$$\Pr\left[\text{Rand}^{A}_{\{0,1\}^3} \Rightarrow \text{true}\right] = 1/8$$

Random Functions

Game $\mathrm{Rand}_{\{0,1\}^3}$

procedure Fn(x)

if $\mathsf{T}[x] = \perp$ then $\mathsf{T}[x] \xleftarrow{\$} \{0,1\}^3$

return $\mathsf{T}[x]$

adversary A

$y_1 \leftarrow \mathbf{Fn}(00)$

$y_2 \leftarrow \mathbf{Fn}(11)$

return $(y_1 = 010 \wedge y_2 = 011)$

$$\Pr\left[\mathrm{Rand}^{A}_{\{0,1\}^3} \Rightarrow \mathsf{true}\right] = \frac{1}{2^6}$$

Random Functions

Game $\mathrm{Rand}_{\{0,1\}^3}$

procedure Fn(x)

if $\mathsf{T}[x] = \perp$ then $\mathsf{T}[x] \xleftarrow{\$} \{0,1\}^3$

return $\mathsf{T}[x]$

adversary A

$y_1 \leftarrow$ **Fn**(00)

$y_2 \leftarrow$ **Fn**(11)

return $(y_1 \oplus y_2 = 101)$

$$\Pr\left[\mathrm{Rand}^{A}_{\{0,1\}^3} \Rightarrow \mathsf{true}\right] = \frac{1}{8}$$

Function Families

$\{F_K\}_{K \in Keys}$

A family of functions $F : \text{Keys}(F) \times \text{Dom}(F) \rightarrow \text{Range}(F)$ is a two-argument map. For $K \in \text{Keys}(F)$ we let $F_K : \text{Dom}(F) \rightarrow \text{Range}(F)$ be defined by

$$\forall x \in \text{Dom}(F) : F_K(x) = F(K, x)$$

Examples:

- DES: $\text{Keys} = \{0,1\}^{56}$, $\text{D} = \text{R} = \{0,1\}^{64}$
- Any block cipher: $\text{D} = \text{R}$ and each F_K is a permutation

Intuition

Notion	Real object	Ideal object
PRF	Family of functions (eg. a block cipher)	Random function

F is a PRF if the input-output behavior of F_K looks to a tester like the input-output behavior of a random function.

Tester does not get the key K!

The Games

Let F: $\text{Keys}(F) \times \text{Dom}(F) \rightarrow \text{Range}(F)$ be a family of functions.

Game Real_F	Game $\text{Rand}_{\text{Range}(F)}$
procedure Initialize $K \xleftarrow{\$} \text{Keys}(F)$ **procedure Fn**(x) Return $F_K(x)$	**procedure Fn**(x) $T[x] \xleftarrow{\$} \text{Range}(F)$ Return $T[x]$

Associated to F, A are the probabilities

$$\Pr\left[\text{Real}_F^A \Rightarrow 1\right] \quad \Big| \quad \Pr\left[\text{Rand}_{\text{Range}(F)}^A \Rightarrow 1\right]$$

that A outputs 1 in each world. The advantage of A is

$$\mathbf{Adv}_F^{\text{prf}}(A) = \Pr\left[\text{Real}_F^A \Rightarrow 1\right] - \Pr\left[\text{Rand}_{\text{Range}(F)}^A \Rightarrow 1\right]$$

PRF advantage

A's output d	Intended meaning: I think I am in game
1	Real
0	Random

$\mathbf{Adv}_F^{\mathrm{prf}}(A) \approx 1$ means A is doing well and F is not prf-secure.
$\mathbf{Adv}_F^{\mathrm{prf}}(A) \approx 0$ (or ≤ 0) means A is doing poorly and F resists the attack A is mounting.

pseudo random function

PRF Security

Func.
Fam.

Adversary advantage depends on its

- strategy
- resources: Running time t and number q of oracle queries

Security: F is a (secure) PRF if $\mathbf{Adv}_F^{\mathrm{prf}}(A)$ is "small" for ALL A that use "practical" amounts of resources.

Example: 80-bit security could mean that for all $n = 1, \ldots, 80$ we have

$$\mathbf{Adv}_F^{\mathrm{prf}}(A) \leq 2^{-n}$$

for any A with time and number of oracle queries at most 2^{80-n}.

Insecurity: F is insecure (not a PRF) if we can specify an A using "few" resources that achieves "high" advantage.

Examples

Define F: $\{0,1\}^\ell \times \{0,1\}^\ell \to \{0,1\}^\ell$ by $F_K(x) = K \oplus x$ for all $K, x \in \{0,1\}^\ell$. Is F a secure PRF?

Game Real_F	Game $\mathrm{Rand}_{\{0,1\}^\ell}$
procedure Initialize $K \xleftarrow{\$} \{0,1\}^\ell$ **procedure Fn**(x) Return $K \oplus x$	**procedure Fn**(x) if $\mathrm{T}[x] = \bot$ then $\mathrm{T}[x] \xleftarrow{\$} \{0,1\}^\ell$ Return $\mathrm{T}[x]$

So we are asking: Can we design a low-resource A so that

$$\mathbf{Adv}_F^{\mathrm{prf}}(A) = \Pr\left[\mathrm{Real}_F^A \Rightarrow 1\right] - \Pr\left[\mathrm{Rand}_{\{0,1\}^\ell}^A \Rightarrow 1\right]$$

is close to 1?

Examples

Exploitable weakness of F: For all K we have

$$F_K(0^\ell) \oplus F_K(1^\ell) = (K \oplus 0^\ell) \oplus (K \oplus 1^\ell) = 1^\ell$$

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F: $\{0,1\}^\ell \times \{0,1\}^\ell \to \{0,1\}^\ell$ is defined by $F_K(x) = K \oplus x$.

adversary A
if $\mathbf{Fn}(0^\ell) \oplus \mathbf{Fn}(1^\ell) = 1^\ell$ then return 1 else return 0

Real game analysis

F: $\{0,1\}^\ell \times \{0,1\}^\ell \to \{0,1\}^\ell$ is defined by $F_K(x) = K \oplus x$.

adversary A

if $\mathbf{Fn}(0^\ell) \oplus \mathbf{Fn}(1^\ell) = 1^\ell$ then return 1 else return 0

Game Real_F

procedure Initialize

$K \xleftarrow{\$} \{0,1\}^\ell$

procedure Fn(x)

Return $K \oplus x$

$\Pr\left[\mathrm{Real}_F^A \Rightarrow 1\right] = 1$

Rand game analysis

F: $\{0,1\}^\ell \times \{0,1\}^\ell \to \{0,1\}^\ell$ is defined by $F_K(x) = K \oplus x$.

adversary A

if $\mathbf{Fn}(0^\ell) \oplus \mathbf{Fn}(1^\ell) = 1^\ell$ then return 1 else return 0

Game $\mathrm{Rand}_{\{0,1\}^\ell}$

procedure Fn(x)

if $\mathsf{T}[x] = \perp$ then $\mathsf{T}[x] \xleftarrow{\$} \{0,1\}^\ell$

Return $\mathsf{T}[x]$

$$\Pr\left[\mathrm{Rand}_{\{0,1\}^\ell}^{A} \Rightarrow 1\right] = \frac{1}{2^\ell}$$

Putting It Together

F: $\{0,1\}^\ell \times \{0,1\}^\ell \to \{0,1\}^\ell$ is defined by $F_K(x) = K \oplus x$.

adversary A
if $\mathbf{Fn}(0^\ell) \oplus \mathbf{Fn}(1^\ell) = 1^\ell$ then return 1 else return 0

Then

$$\begin{aligned}\mathbf{Adv}_F^{\mathrm{prf}}(A) &= \overbrace{\Pr\left[\mathrm{Real}_F^A \Rightarrow 1\right]}^{1} - \overbrace{\Pr\left[\mathrm{Rand}_{\{0,1\}^\ell}^A \Rightarrow 1\right]}^{2^{-\ell}} \\ &= 1 - 2^{-\ell}\end{aligned}$$

and A is efficient .

Conclusion: F is not a secure PRF.

Blockciphers as PRFs

Let $E\colon \{0,1\}^k \times \{0,1\}^\ell \to \{0,1\}^\ell$ be a block cipher.

Game Real_E	Game $\mathrm{Rand}_{\{0,1\}^\ell}$
procedure Initialize	**procedure Fn**(x)
$K \xleftarrow{\$} \{0,1\}^k$	if $\mathsf{T}[x] = \perp$ then $\mathsf{T}[x] \xleftarrow{\$} \{0,1\}^\ell$
procedure Fn(x)	Return $\mathsf{T}[x]$
Return $E_K(x)$	

Can we design A so that

$$\mathbf{Adv}^{\mathrm{prf}}_E(A) = \Pr\left[\mathrm{Real}^A_E \Rightarrow 1\right] - \Pr\left[\mathrm{Rand}^A_{\{0,1\}^\ell} \Rightarrow 1\right]$$

is close to 1?

Generic Attacks on blockciphers as PRFs

Generic Attacks on blockciphers as PRFs

Exhaustive Key Search Attack

← ad vantange proportional to key length

Generic Attacks on blockciphers as PRFs

Generic Attacks on blockciphers as PRFs

Birthday Attack — advantage proportional to block-length

Birthday Attack

We have q people $1, \ldots, q$ with birthdays $y_1, \ldots, y_q \in \{1, \ldots, 365\}$. Assume each person's birthday is a random day of the year. Let

$$\begin{aligned} C(365, q) &= \Pr[\text{2 or more persons have same birthday}] \\ &= \Pr[y_1, \ldots, y_q \text{ are not all different}] \end{aligned}$$

- What is the value of $C(365, q)$?
- How large does q have to be before $C(365, q)$ is at least $1/2$?

Naive intuition:

- $C(365, q) \approx q/365$
- q has to be around 365

The reality

- $C(365, q) \approx q^2/365$
- q has to be only around 23

Birthday Collision Bounds

$C(365, q)$ is the probability that some two people have the same birthday in a room of q people with random birthdays

q	$C(365, q)$
15	0.253
18	0.347
20	0.411
21	0.444
23	0.507
25	0.569
27	0.627
30	0.706
35	0.814
40	0.891
50	0.970

Birthday problem

Coll(N, q)

Pick $y_1, \ldots, y_q \xleftarrow{\$} \{1, \ldots, N\}$ and let

$$C(N,q) = \Pr[y_1, \ldots, y_q \textbf{ not } \text{all distinct}]$$

Birthday setting: $N = 365$

Fact: $C(N,q) \approx \frac{q^2}{2N}$

want good upper & lower-bounds on C(N, q).

Upper-bound: let $COLL_i$ be the event that there's a collision when i-th element y_i is chosen.

$$C(n,q) \leq \Pr[\bigvee_i COLL_i] \leq \sum_i \Pr[COLL_i]$$

Birthday collision formula

Let $y_1, \ldots, y_q \stackrel{\$}{\leftarrow} \{1, \ldots, N\}$. Then

$$
\begin{aligned}
1 - C(N, q) &= \Pr[y_1, \ldots, y_q \text{ all distinct}] \\
&= 1 \cdot \frac{N-1}{N} \cdot \frac{N-2}{N} \cdot \cdots \cdot \frac{N-(q-1)}{N} \\
&= \prod_{i=1}^{q-1} \left(1 - \frac{i}{N}\right)
\end{aligned}
$$

so

$$C(N, q) = 1 - \prod_{i=1}^{q-1} \left(1 - \frac{i}{N}\right)$$

$$1 - x \le e^{-x}$$

$$1 - e^{-q(q-1)/2}$$

Birthday bounds

Let

$$C(N, q) = \Pr\left[y_1, \ldots, y_q \textbf{ not} \text{ all distinct}\right]$$

Fact: Then

$$0.3 \cdot \frac{q(q-1)}{N} \leq C(N, q) \leq 0.5 \cdot \frac{q(q-1)}{N}$$

where the lower bound holds for $1 \leq q \leq \sqrt{2N}$.

comes from an inequality applied to get the estimate.

Birthday attack adversary

Defining property of a block cipher: E_K is a permutation for every K

So if $x_1, \ldots, x_q$ are distinct then

- $\mathbf{Fn} = E_K \Rightarrow \mathbf{Fn}(x_1), \ldots, \mathbf{Fn}(x_q)$ distinct
- $\mathbf{Fn}$ random $\Rightarrow \mathbf{Fn}(x_1), \ldots, \mathbf{Fn}(x_q)$ not necessarily distinct

This leads to the following attack:

adversary A

Let $x_1, \ldots, x_q \in \{0,1\}^\ell$ be distinct
for $i = 1, \ldots, q$ do $y_i \leftarrow \mathbf{Fn}(x_i)$
if $y_1, \ldots, y_q$ are all distinct then return 1
else return 0

Real game analysis

Let $E: \{0,1\}^k \times \{0,1\}^\ell \rightarrow \{0,1\}^\ell$ be a block cipher

Game Real_E

procedure Initialize

$K \stackrel{\$}{\leftarrow} \{0,1\}^k$

procedure Fn(x)

Return $E_K(x)$

adversary A

Let $x_1, \ldots, x_q \in \{0,1\}^\ell$ be distinct

for $i = 1, \ldots, q$ do $y_i \leftarrow \mathbf{Fn}(x_i)$

if $y_1, \ldots, y_q$ are all distinct

then return 1 else return 0

Then

$$\Pr\left[\mathrm{Real}_E^A \Rightarrow 1\right] = 1$$

Rand game analysis

Let $E: \{0,1\}^K \times \{0,1\}^\ell \to \{0,1\}^\ell$ be a block cipher

Game $\mathrm{Rand}_{\{0,1\}^\ell}$
procedure Fn(x)
if $\mathsf{T}[x] = \perp$ then $\mathsf{T}[x] \xleftarrow{\$} \{0,1\}^\ell$
Return $\mathsf{T}[x]$

adversary A

Let $x_1, \ldots, x_q \in \{0,1\}^\ell$ be distinct
for $i = 1, \ldots, q$ do $y_i \leftarrow \mathbf{Fn}(x_i)$
if $y_1, \ldots, y_q$ are all distinct
then return 1 else return 0

Then

$$\Pr\left[\mathrm{Rand}^A_{\{0,1\}^\ell} \Rightarrow 1\right] = \Pr\left[y_1, \ldots, y_q \text{ all distinct}\right] = 1 - C(2^\ell, q)$$

because $y_1, \ldots, y_q$ are randomly chosen from $\{0,1\}^\ell$.

Birthday attack conclusion

$E : \{0,1\}^k \times \{0,1\}^\ell \rightarrow \{0,1\}^\ell$ a block cipher

adversary A

Let $x_1, \ldots, x_q \in \{0,1\}^\ell$ be distinct
for $i = 1, \ldots, q$ do $y_i \leftarrow \mathbf{Fn}(x_i)$
if $y_1, \ldots, y_q$ are all distinct then return 1 else return 0

$$\mathbf{Adv}_E^{\mathrm{prf}}(A) = \overbrace{\Pr\left[\mathrm{Real}_E^A \Rightarrow 1\right]}^{1} - \overbrace{\Pr\left[\mathrm{Rand}_{\{0,1\}^\ell}^A \Rightarrow 1\right]}^{1 - C(2^\ell, q)}$$

$$= C(2^\ell, q) \geq 0.3 \cdot \frac{q(q-1)}{2^\ell}$$

so

$$q \approx 2^{\ell/2} \Rightarrow \mathbf{Adv}_E^{\mathrm{prf}}(A) \approx 1 .$$

Conclusion: If $E : \{0,1\}^k \times \{0,1\}^\ell \to \{0,1\}^\ell$ is a block cipher, there is an attack on it as a PRF that succeeds in about $2^{\ell/2}$ queries.

Depends on block length, not key length!

	ℓ	$2^{\ell/2}$	Status
DES, 2DES, 3DES3	64	2^{32}	Insecure
AES	128	2^{64}	Secure

pseudo random function

PRP vs PRF

Let F: Keys(F) $\times$ Dom(F) $\rightarrow$ Range(F) be a family of functions.

Game Real_F	Game $\mathrm{Rand}_{\mathsf{Range}(F)}$
procedure Initialize $K \xleftarrow{\$}$ Keys(F) **procedure Fn**(x) Return $F_K(x)$	**procedure Fn**(x) T[x] $\xleftarrow{\$}$ Range(F) { points already in T } Return T[x]

Associated to F, A are the probabilities

$$\Pr\left[\mathrm{Real}_F^A \Rightarrow 1\right] \quad \Big| \quad \Pr\left[\mathrm{Rand}_{\mathsf{Range}(F)}^A \Rightarrow 1\right]$$

that A outputs 1 in each world. The advantage of A is

$$\mathbf{Adv}_F^{\mathrm{prf}}(A) = \Pr\left[\mathrm{Real}_F^A \Rightarrow 1\right] - \Pr\left[\mathrm{Rand}_{\mathsf{Range}(F)}^A \Rightarrow 1\right]$$

Why do we use PRF?

$PRG(K) = E_k(\langle 1 \rangle) \ldots E_k(\langle n \rangle)$

pseudo OTP

start w/ PRP → apply PRP/PRF switching lemma.

need this to look random!

PRF-Security Implications

PRF-security can be seen as a "master property" for blockciphers that implies all other security properties we want.

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E.g., we can show that PRF-security implies security against key-recovery.

KR security vs PRF security

We have seen two possible metrics of security for a block cipher E

- (T)KR-security: It should be hard to find the target key, or a key consistent with input-output examples of a hidden target key.
- PRF-security: It should be hard to distinguish the input-output behavior of E_K from that of a random function.

Fact: PRF-security of E implies

- KR (and hence TKR) security of E
- Many other security attributes of E

This is a validation of the choice of PRF security as our main metric.

Reduction

WTS if $\exists$ adversary A s.t.

$Adv_E^{kr}(A)$ is large then $\exists$

adversary B s.t. $Adv_E^{prf}(A)$ is large.

Conclusion

- We believe DES, AES are "good" blockciphers in the sense that there is no significantly "better than generic" attacks under the PRF notion.

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- Generic attacks:
  - Exhaustive key-search.
  - Birthday attack.

Exercise

We are given a PRF $F\colon \{0,1\}^k \times \{0,1\}^k \to \{0,1\}^k$ and want to build a PRF $G\colon \{0,1\}^k \times \{0,1\}^k \to \{0,1\}^{2k}$. Which of the following work?

1. Function $G(K, x)$
   $y_1 \leftarrow F(K, x)$; $y_2 \leftarrow F(K, \overline{x})$; Return $y_1 \| y_2$

2. Function $G(K, x)$
   $y_1 \leftarrow F(K, x)$; $y_2 \leftarrow F(K, y_1)$; Return $y_1 \| y_2$

3. Function $G(K, x)$
   $L \leftarrow F(K, x)$; $y_1 \leftarrow F(L, 0^k)$; $y_2 \leftarrow F(L, 1^k)$; Return $y_1 \| y_2$

4. Function $G(K, x)$
   [Your favorite code here]