Sampling, Sketching, Streaming, Small-Space Optimization: Algorithmic Approaches *for* Analyzing Large Graphs

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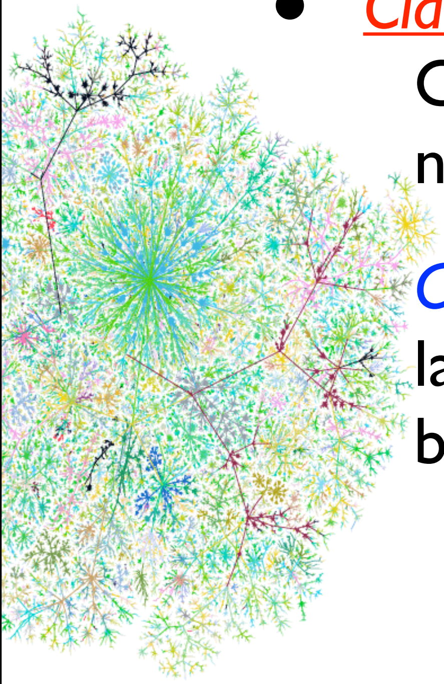
Amazon

Andrew McGregor
University of Massachusetts, Amherst



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  Call graph, web graph, IP graph, social networks, citation networks, protein interaction and metabolic networks....

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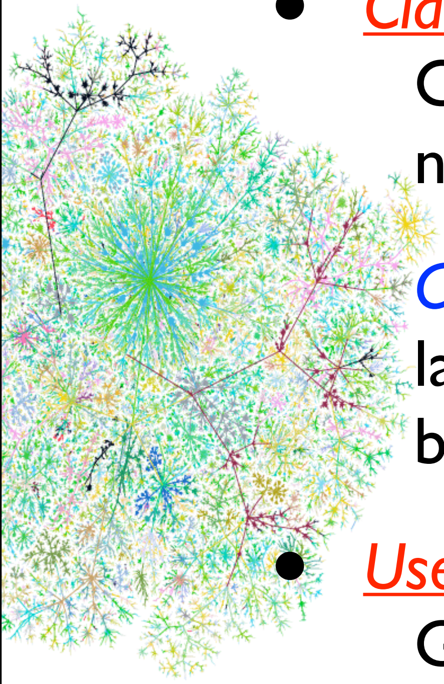

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- Want streaming, parallel, distributed algorithms…

- ***Tutorial Goals and Caveats***

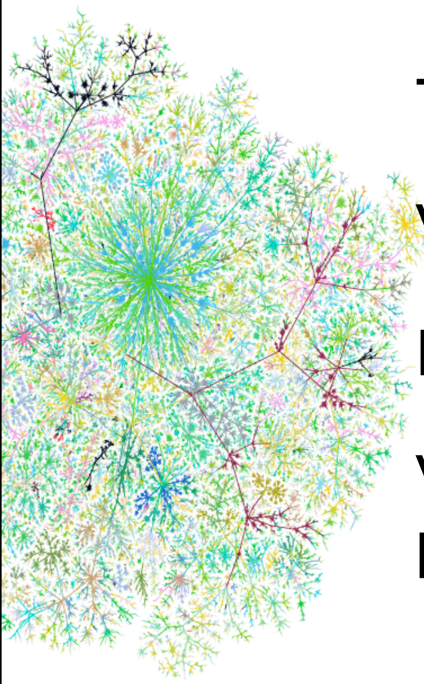
Present some new algorithmic primitives for large graphs.

Techniques are widely applicable; we'll be platform agnostic.

Won't be comprehensive; will cherry pick illustrative results.

Focus on arbitrary graphs rather than specific applications.

Won't focus on proofs but will give basic outline when it helps convey why certain approaches are effective.



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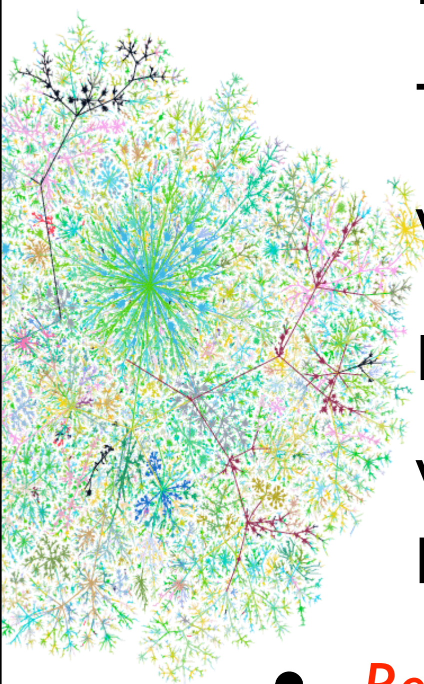


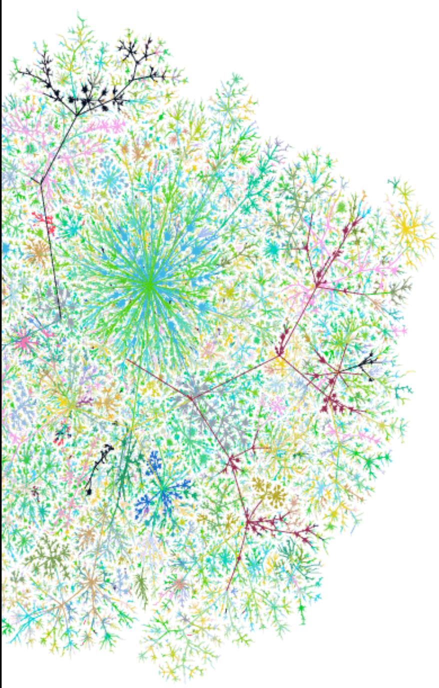
- ***Resources***

Survey: SIGMOD Record
http://people.cs.umass.edu/~mcgregor/papers/graphsurvey.pdf

Tutorial: Slides and Bibliography
http://people.cs.umass.edu/~mcgregor/graphs

Lectures: Ten Lectures on Graph Streams
https://people.cs.umass.edu/~mcgregor/courses/CS711S18/

Overview



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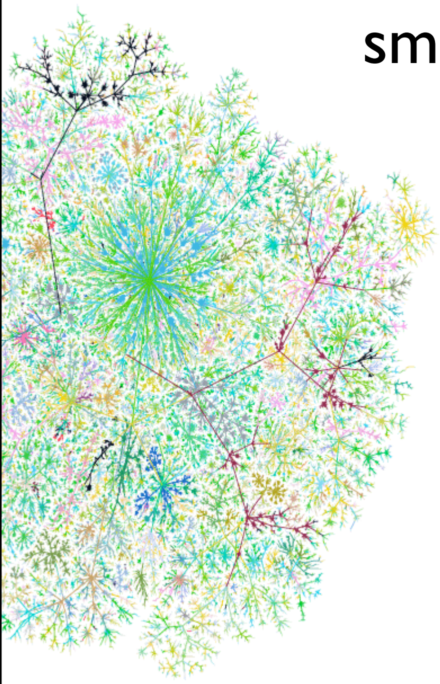
- *Part 1: Sampling* Sampling for finding densest subgraphs, small matchings, triangles, spectral properties.

  "Different sampling techniques for different problems"



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- *Part IV: Small-Space Optimization* Combining sparsification and multiplicative weights for fast, small-space optimization. Examples include large matching and correlation clustering.

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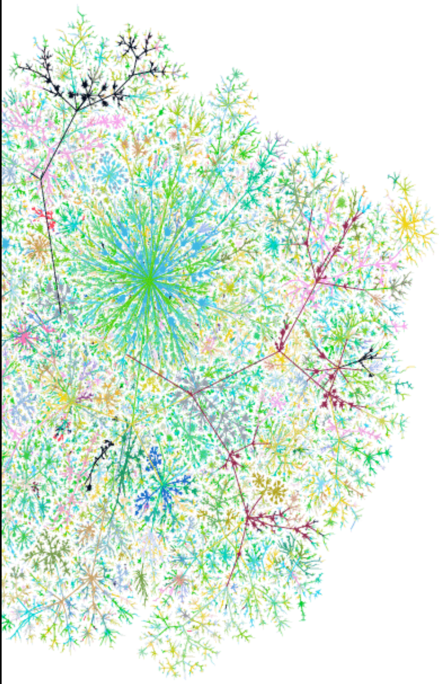

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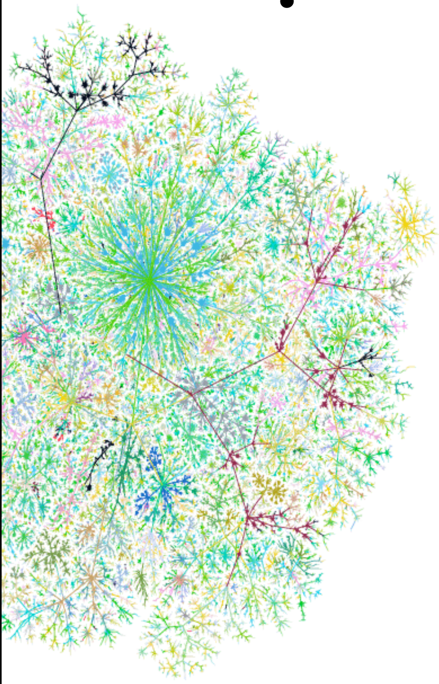

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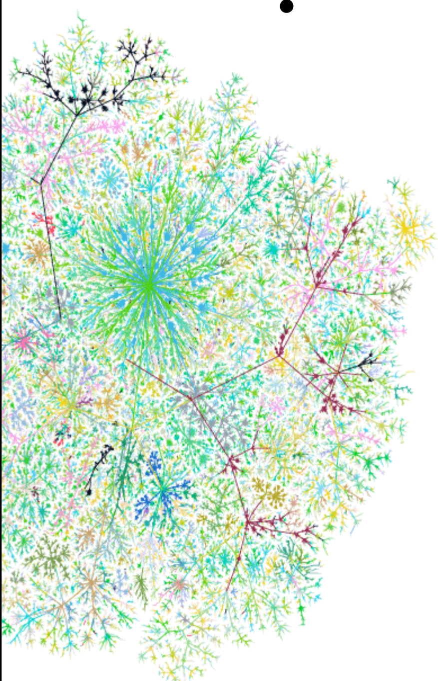

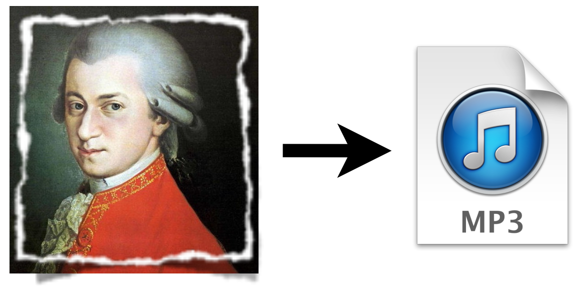

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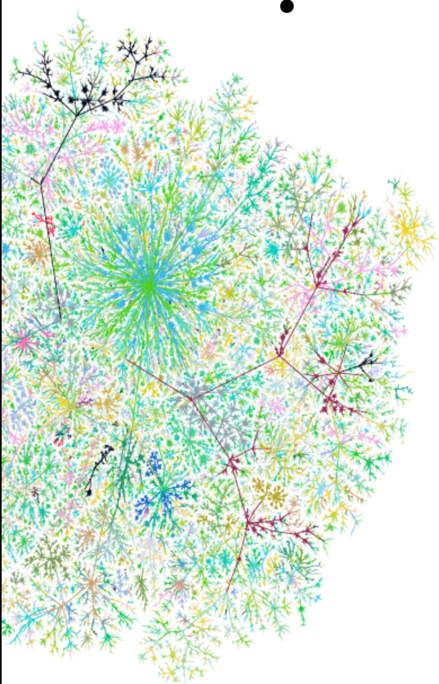

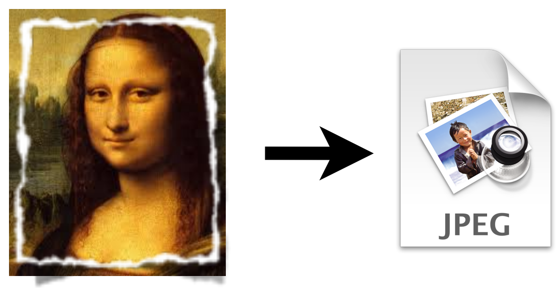

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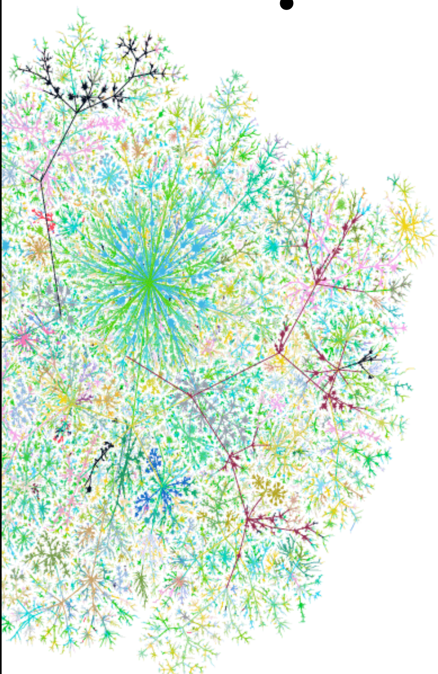

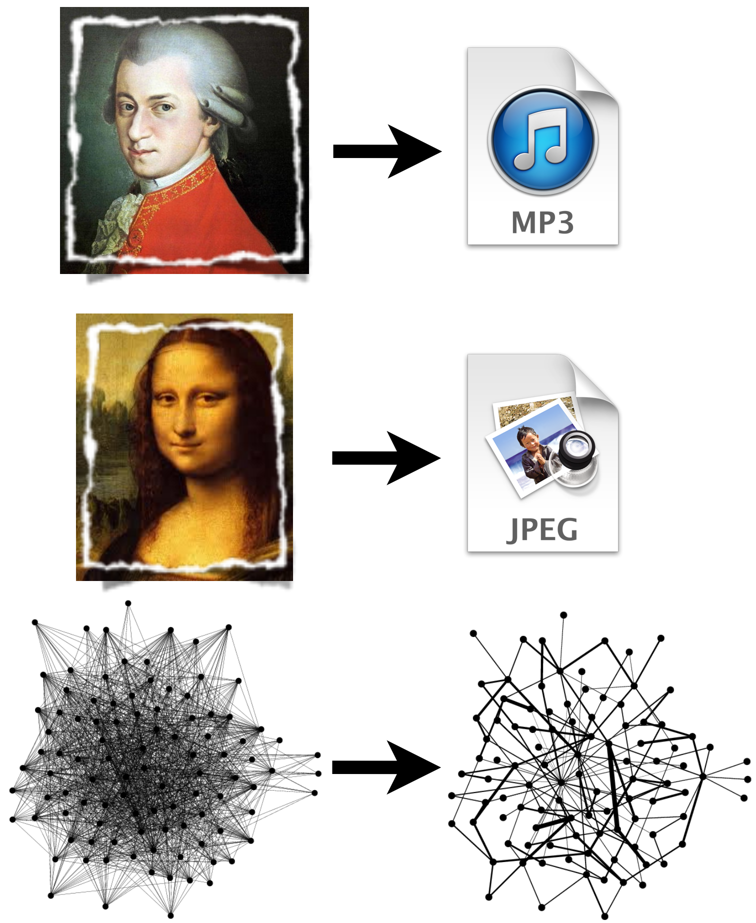

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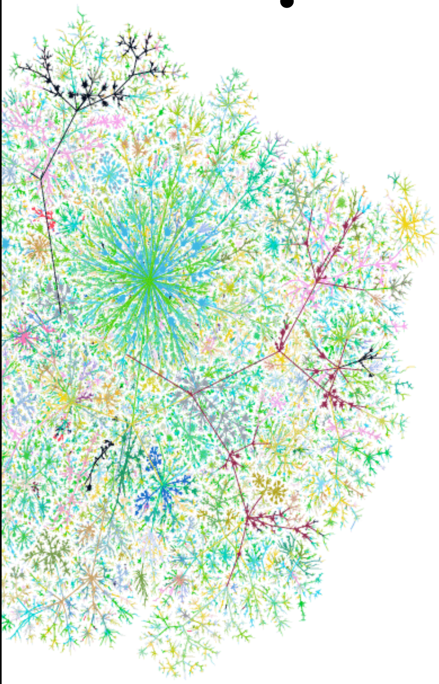
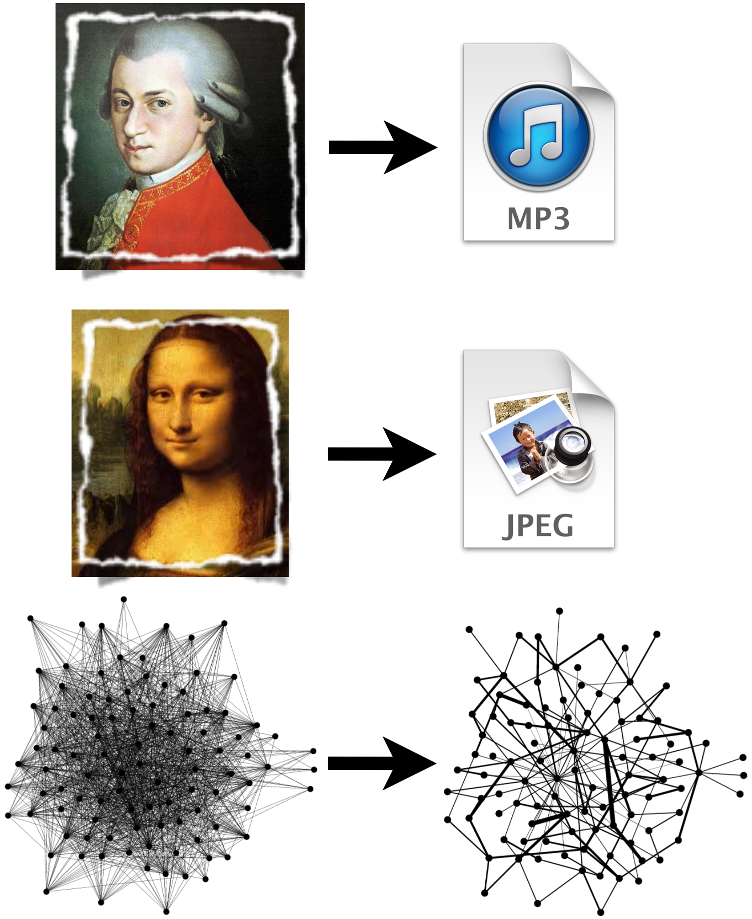
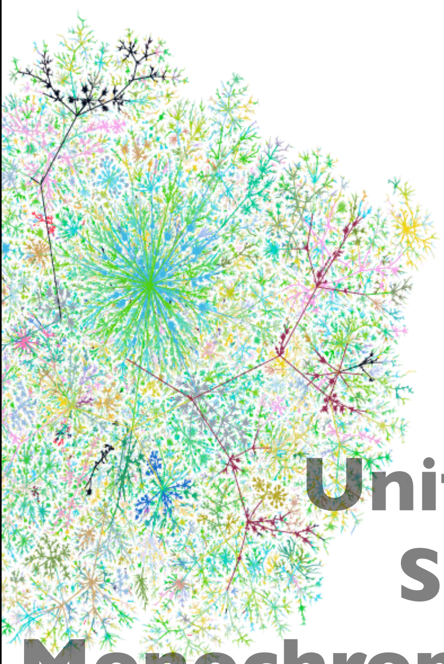




- If compression is easy, we get faster and more-space efficient algorithms by using existing algorithms on compressed graphs.



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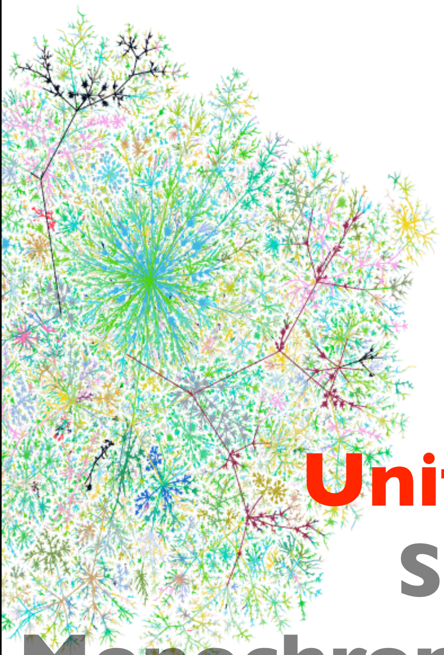

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- *Thm* Sample of $\tilde{O}(\varepsilon^{-2} n)$ edges uniformly and find the densest subgraph in sampled graph. Gives a $(1+\varepsilon)$-approx whp.

McGregor et al. [MFCS 15], *Esfandiari et al.* [SPAA 16]

Mitzenmacher et al. [KDD 15]

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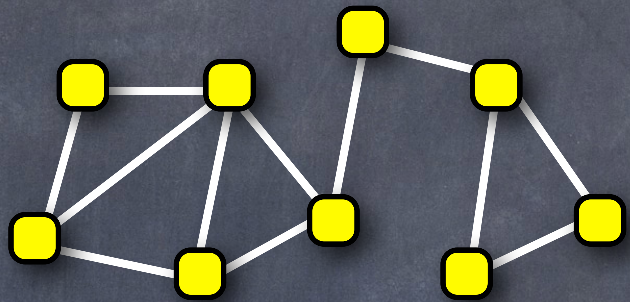

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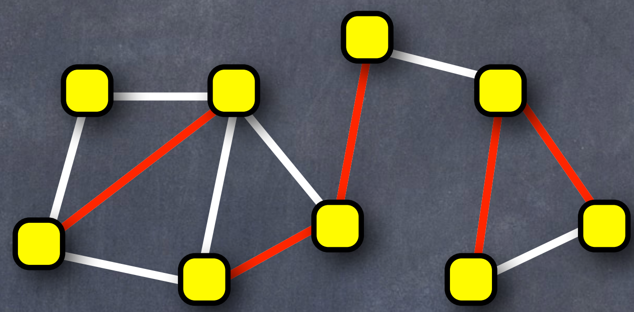

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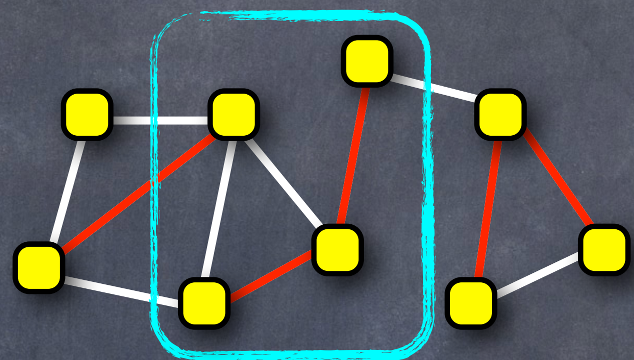

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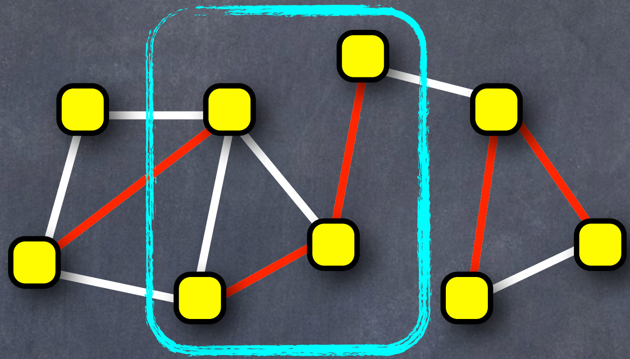

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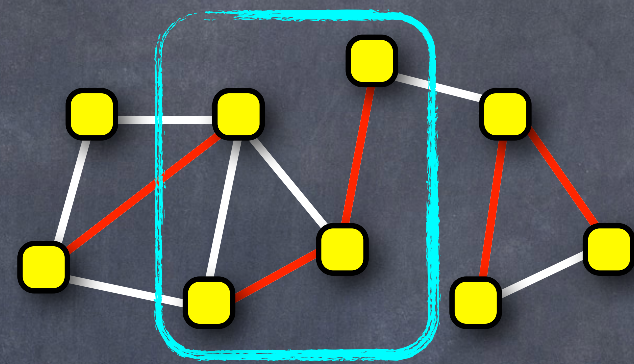

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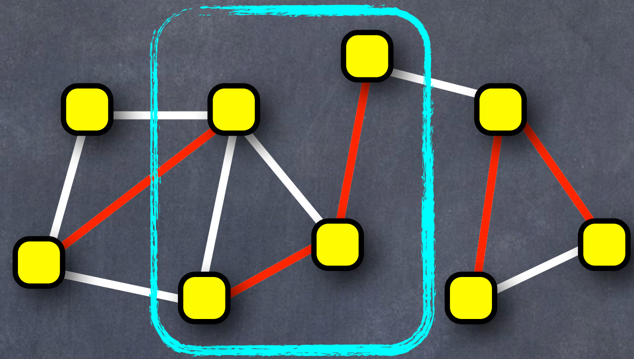

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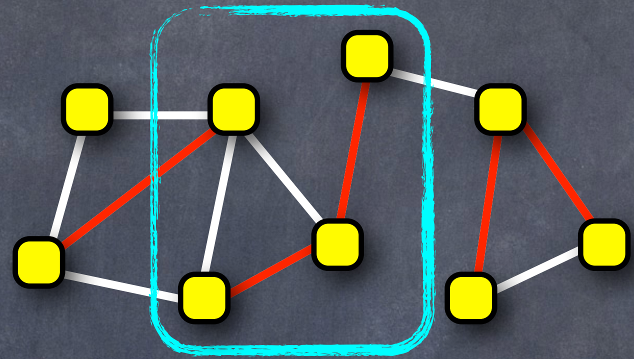

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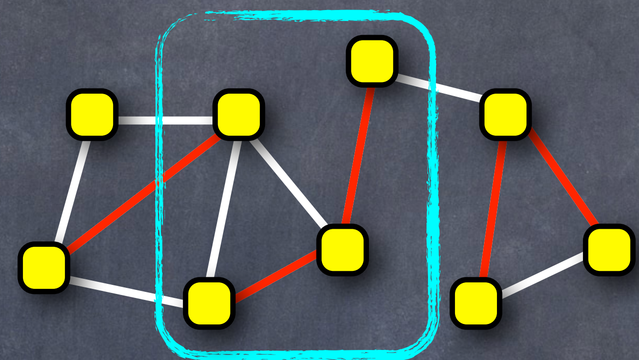

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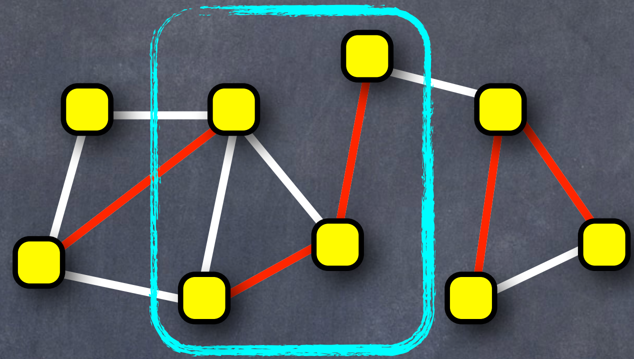

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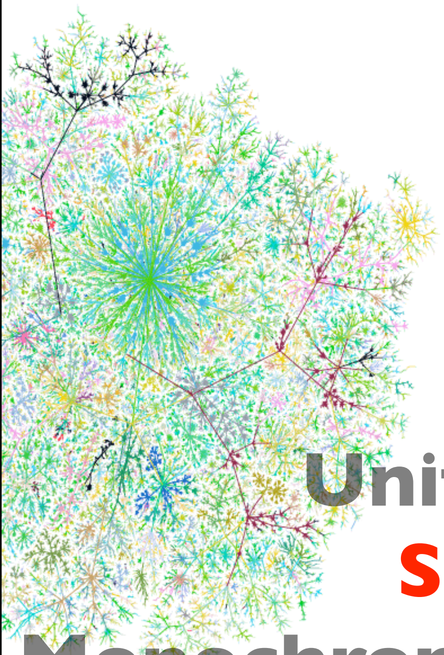

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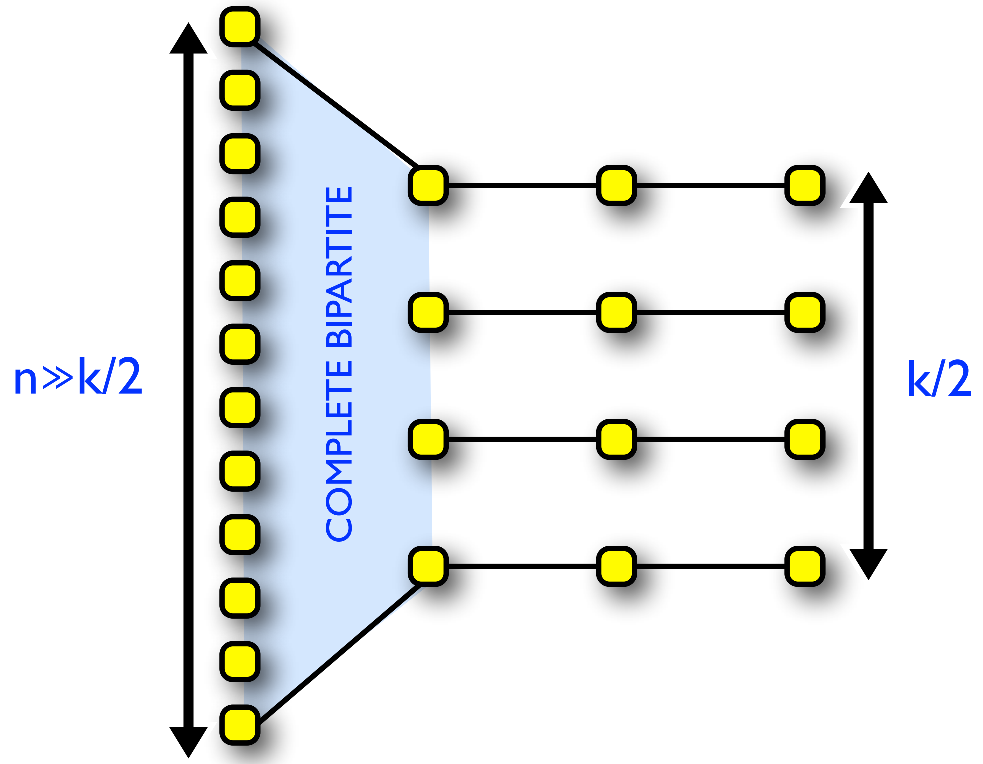

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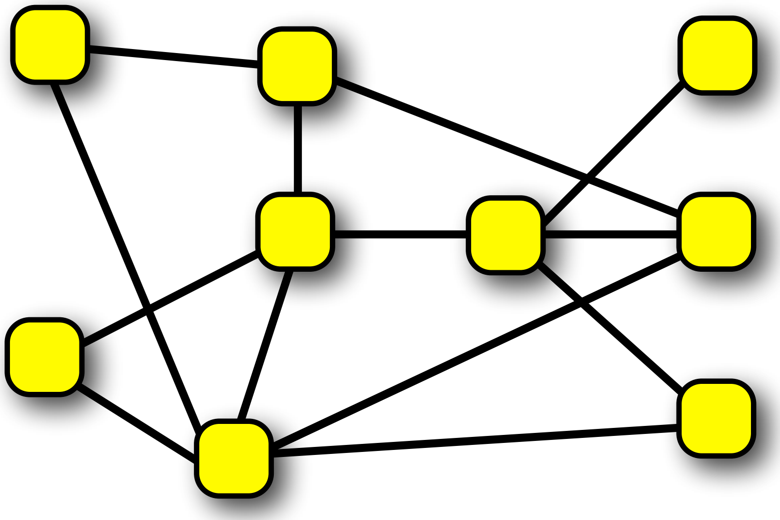

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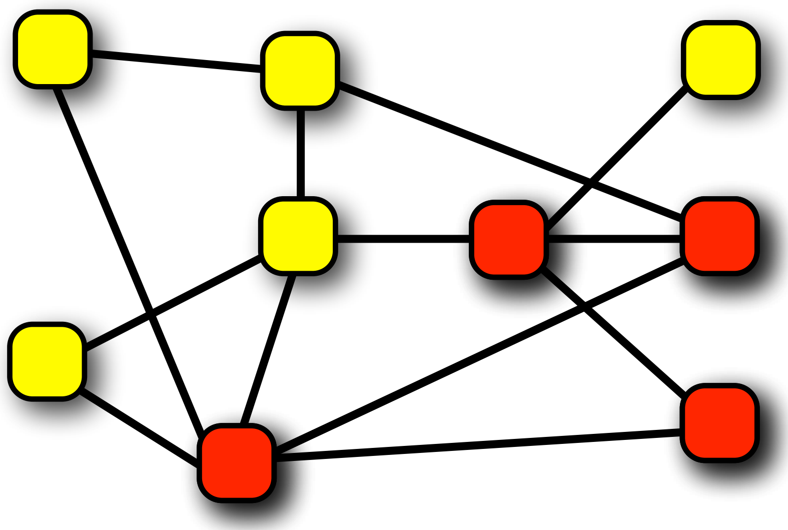

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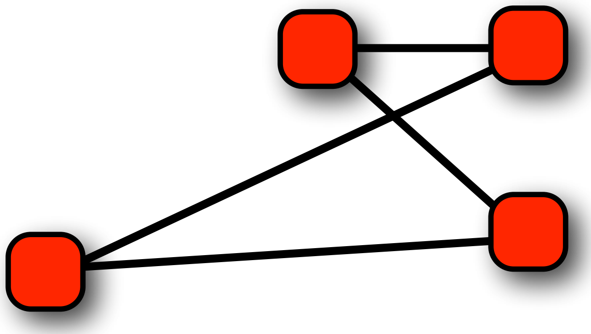

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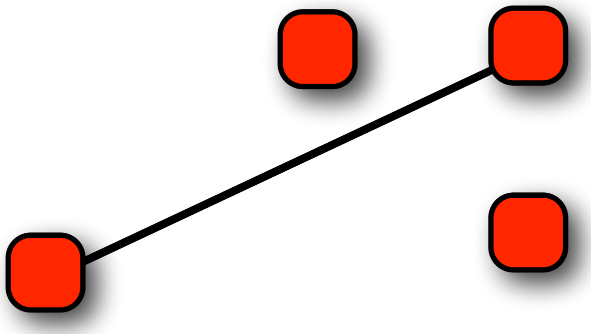

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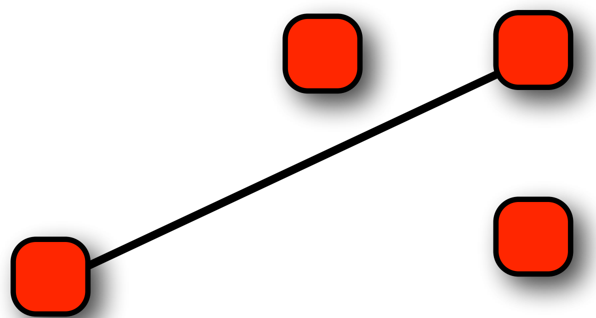

- *Theorem* If G has max matching size k, then $O(k^2 \log k)$ SNAPE samples will include a max matching from G.

Chitnis et al. [SODA 16], *Bury, Schwiegelshohn* [ESA 15]

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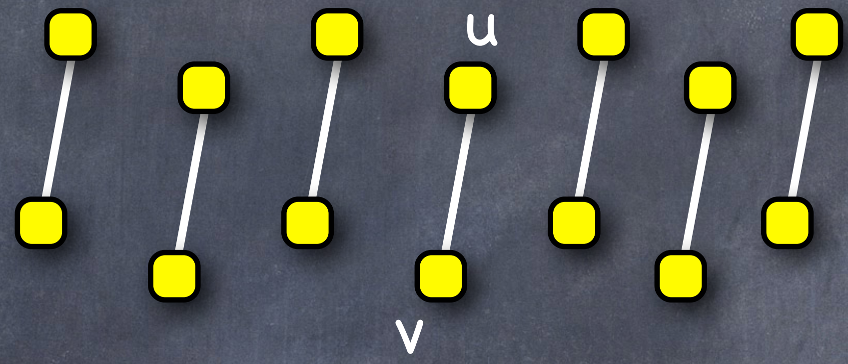

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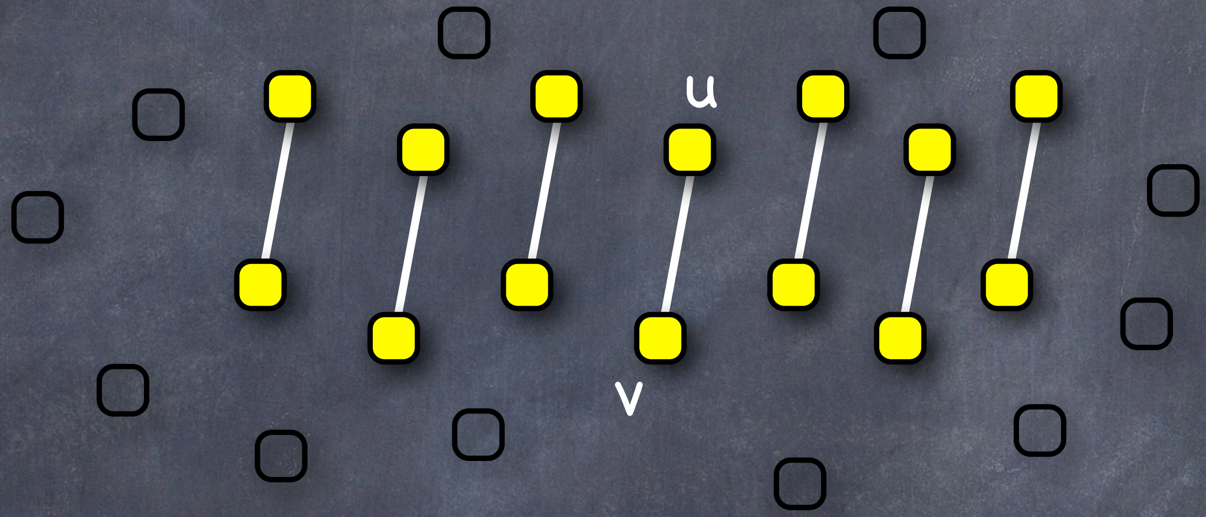

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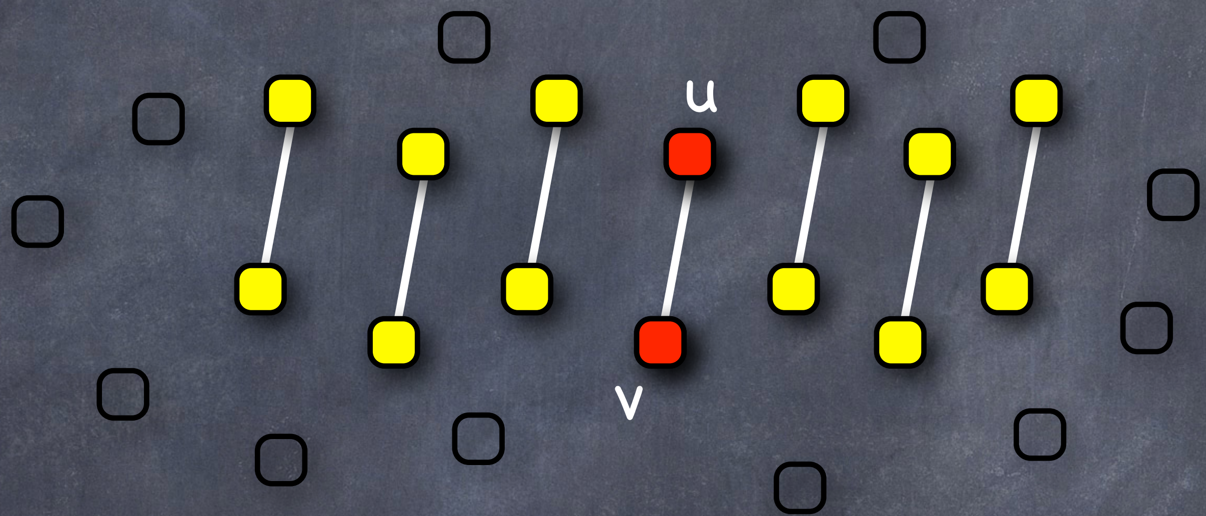

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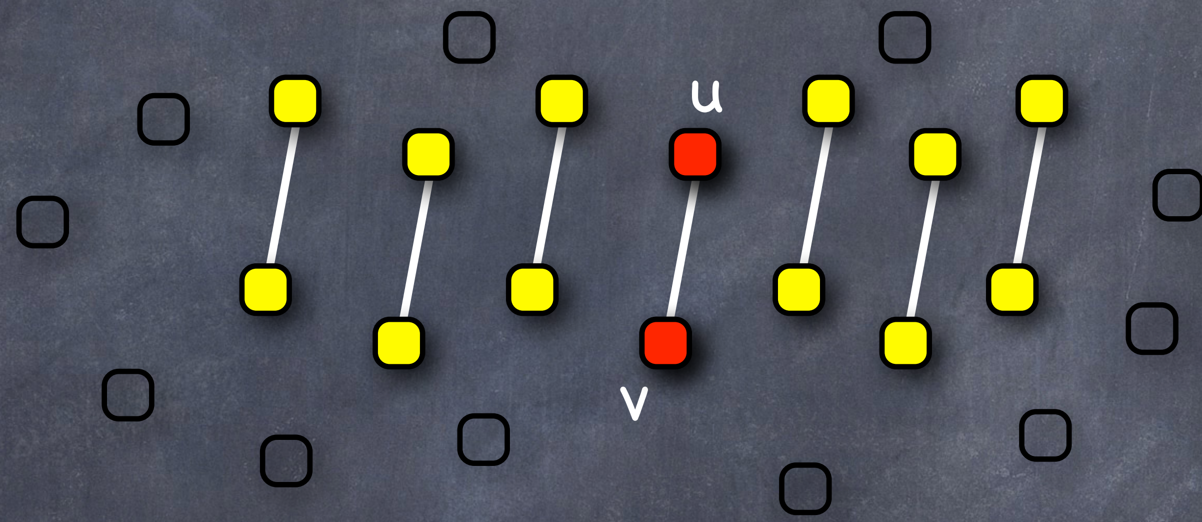

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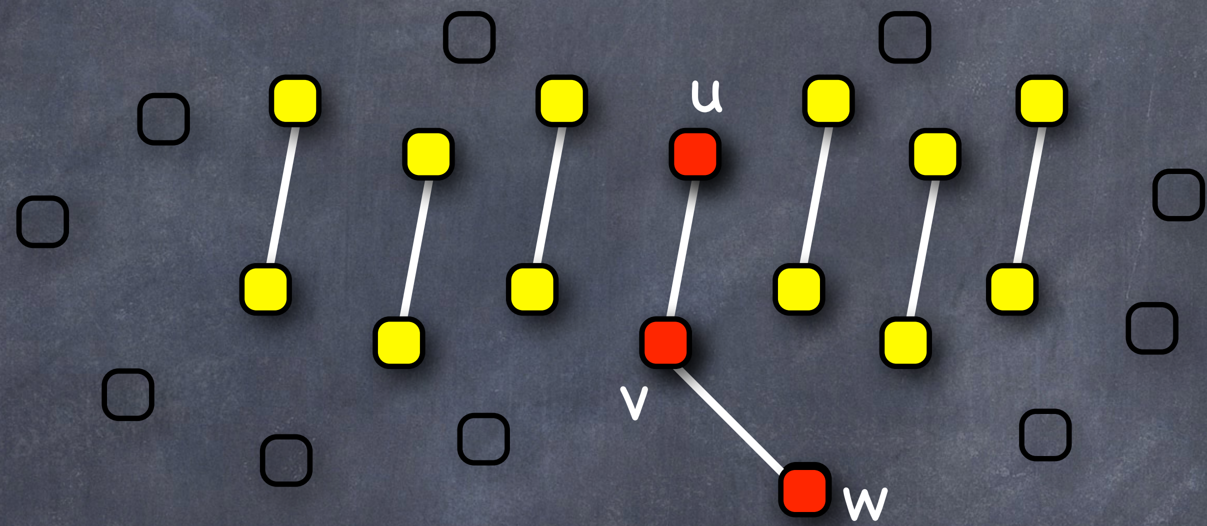

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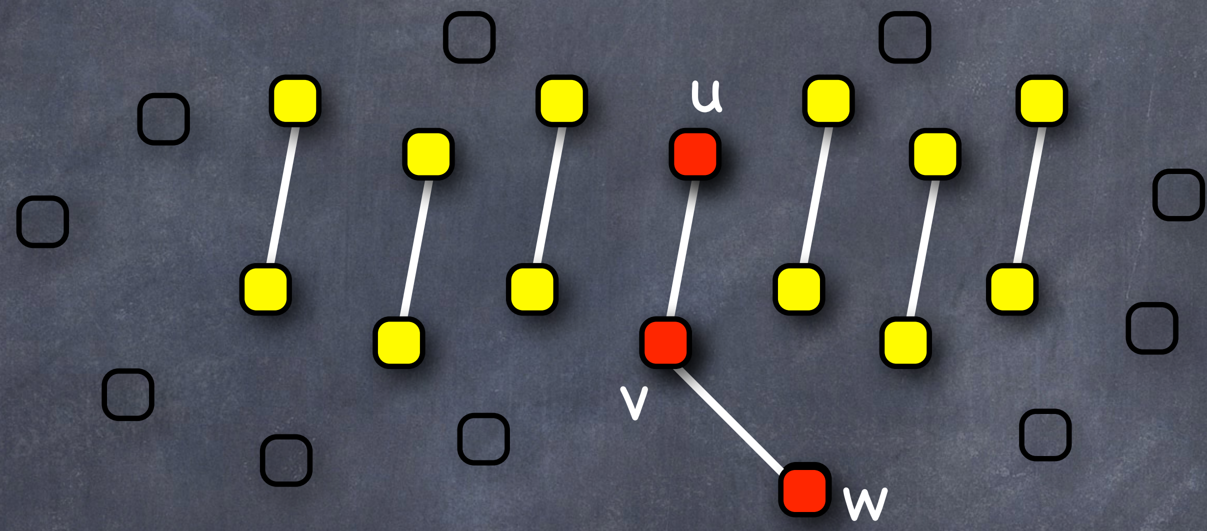

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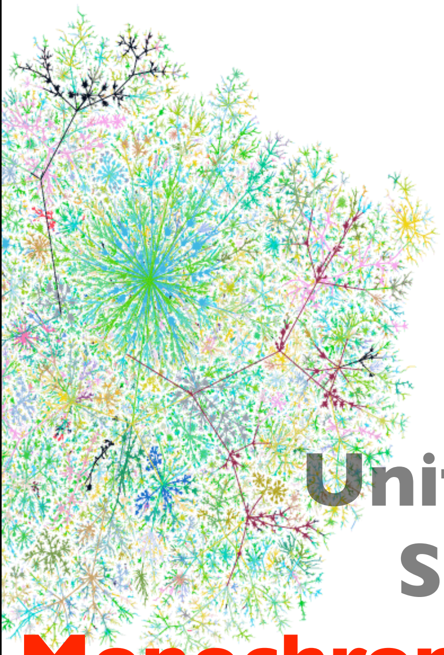

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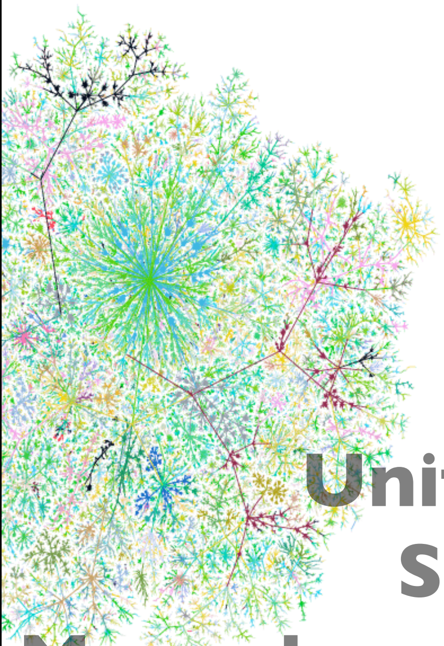
- *Proof Idea* Compute expectation and variance of number of triangles amongst sampled edges and apply Chebyshev bound.



Part I

Sampling

Uniform Sampling + Densest Subgraph

Snape Sampling + Matching

Monochromatic Sampling + Clustering Coefficient

Edge-Weighted Sampling + Cuts and Sparsification

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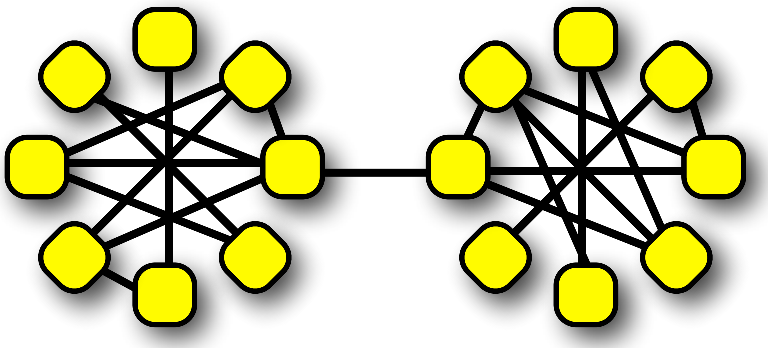

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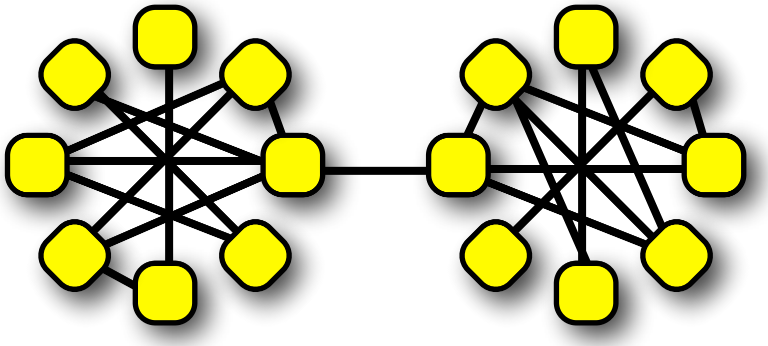

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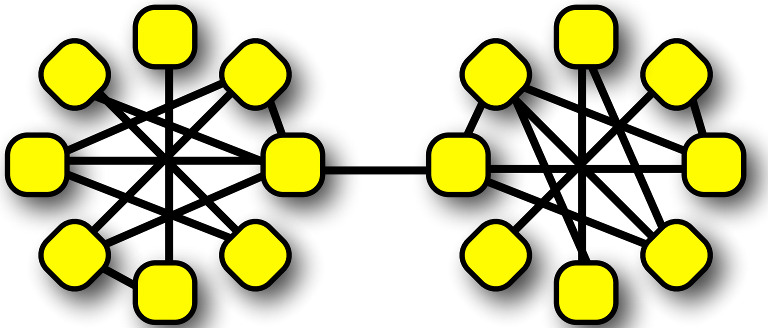

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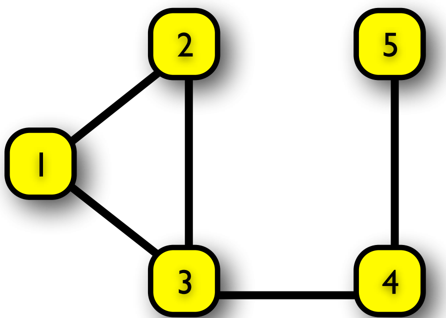

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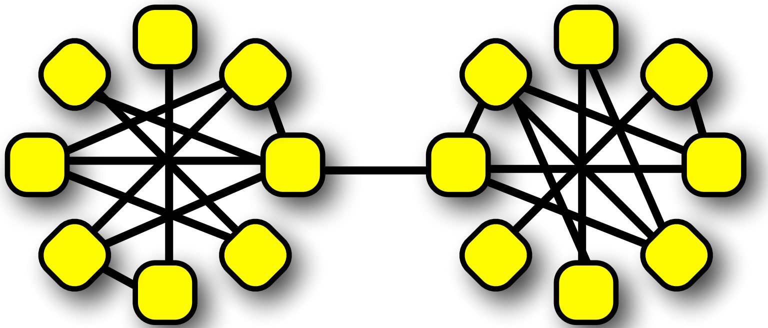

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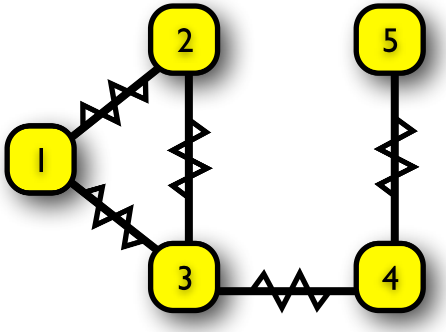

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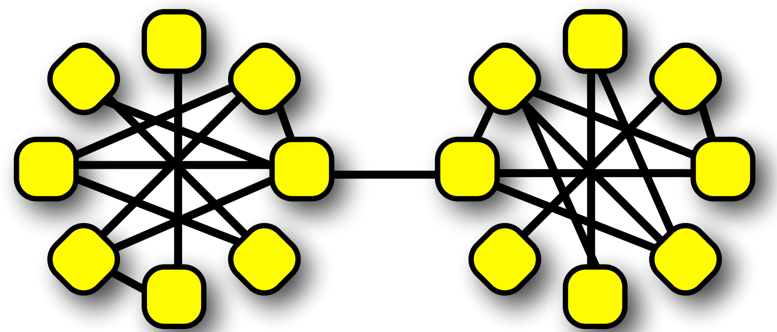

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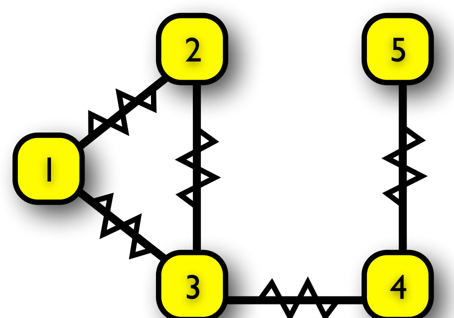

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- *Simpler Thm* If min-cut is $\gg \varepsilon^{-2}\log n$ then $p_e=1/2$ works.

Proof Idea of Simpler Theorem ...

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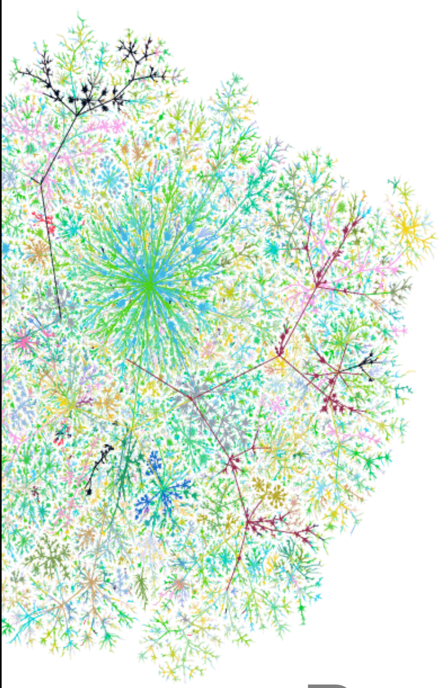
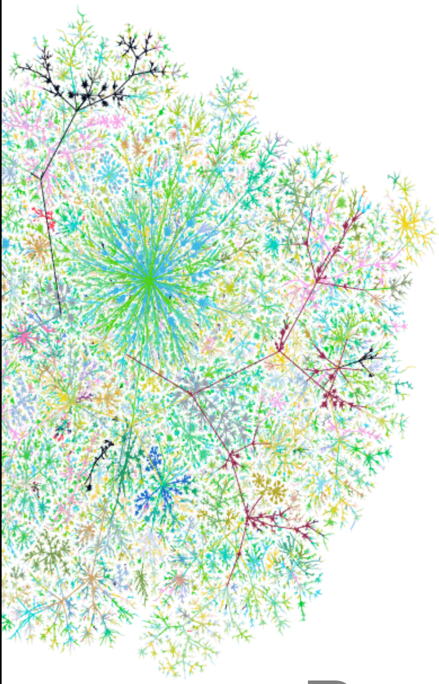
- Result then follows by substituting bound for λ and applying union bound over all cuts.



Part II

Sketching

What is sketching?
Surprising connectivity example
Revisiting graph cuts and sparsification



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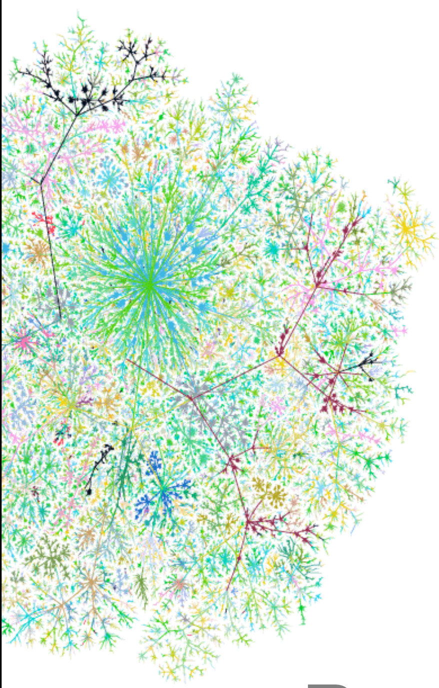
? ***Question*** What about analyzing massive graphs via sketches?



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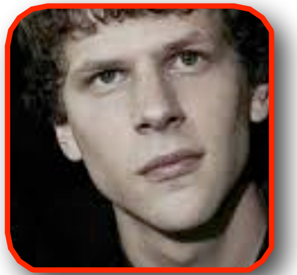
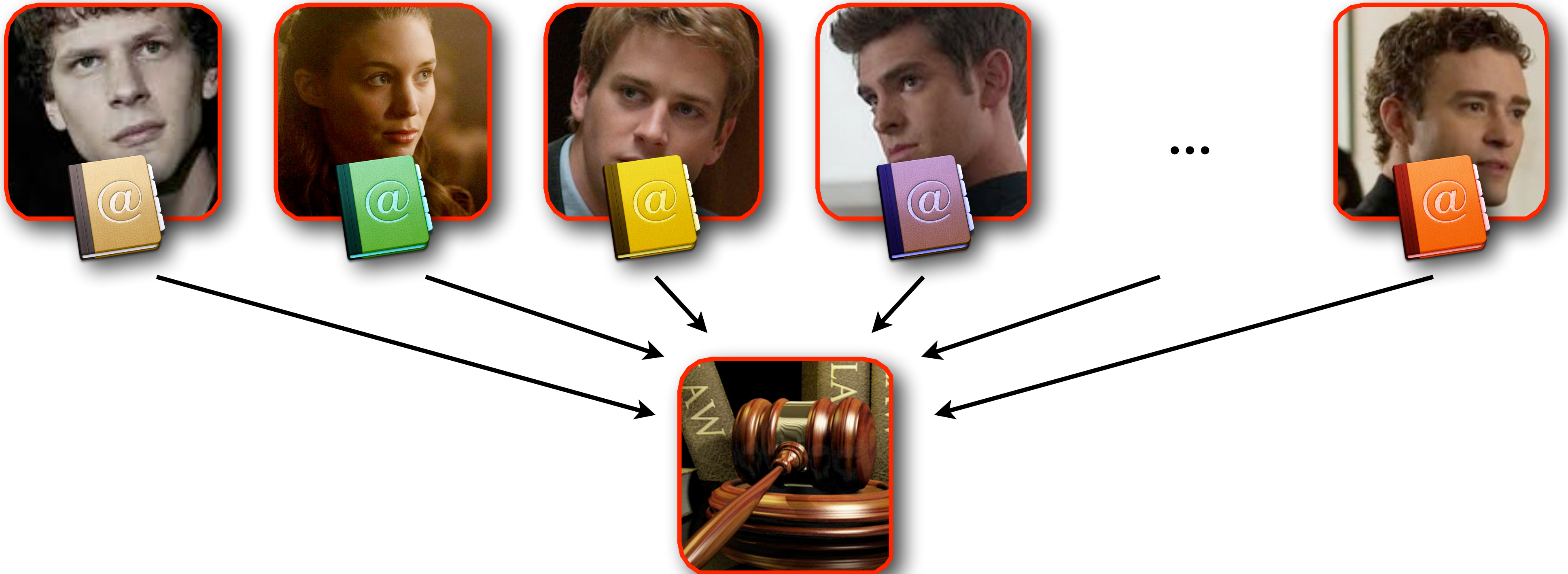
    ... 

- *Communication Problem* n players each have a list their friends. Simultaneously, they each send a message to a central player who deduces if underlying graph is connected.



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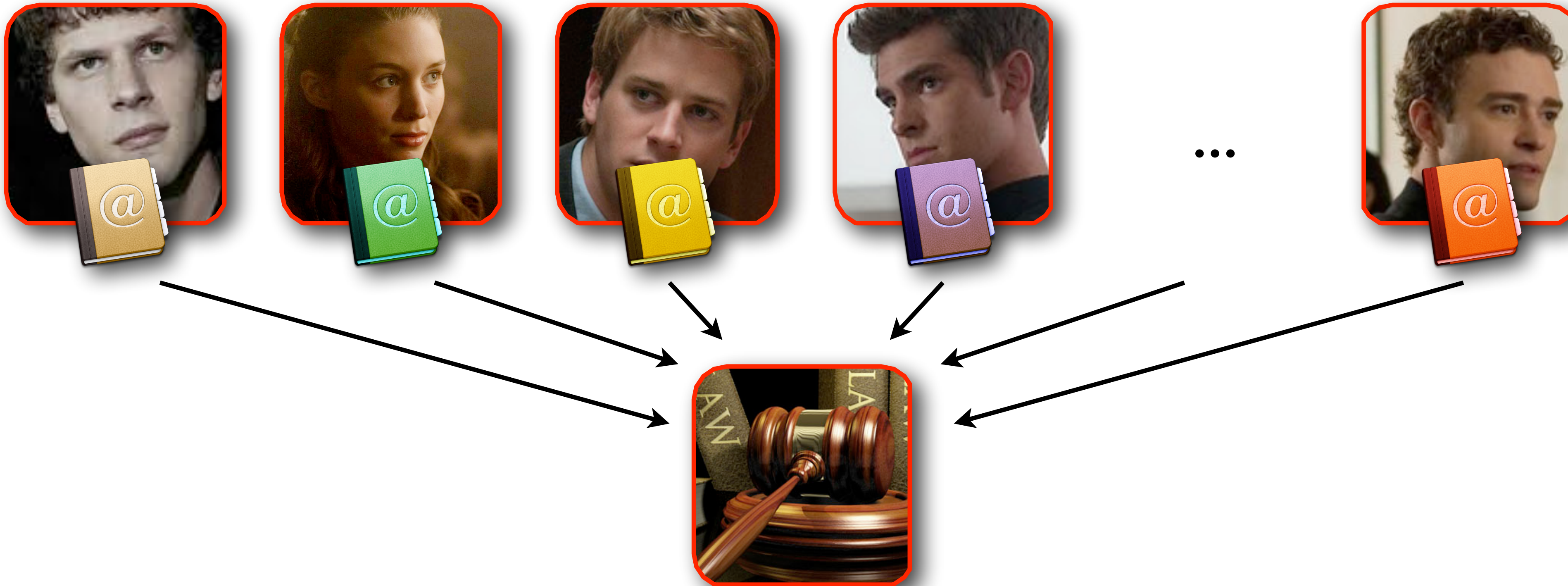

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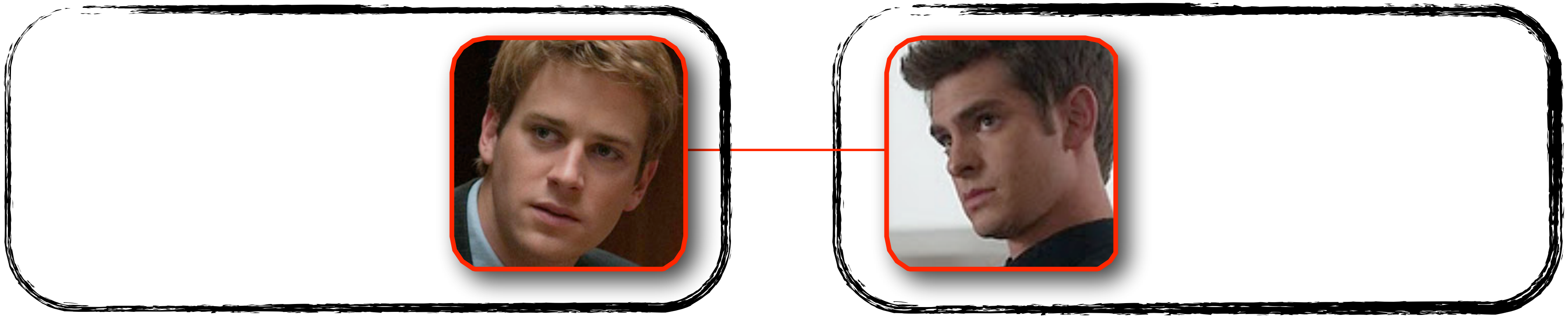
- *Thm* O(polylog n) bit message from each player suffices.

Ahn, Guha, McGregor [SODA 12]



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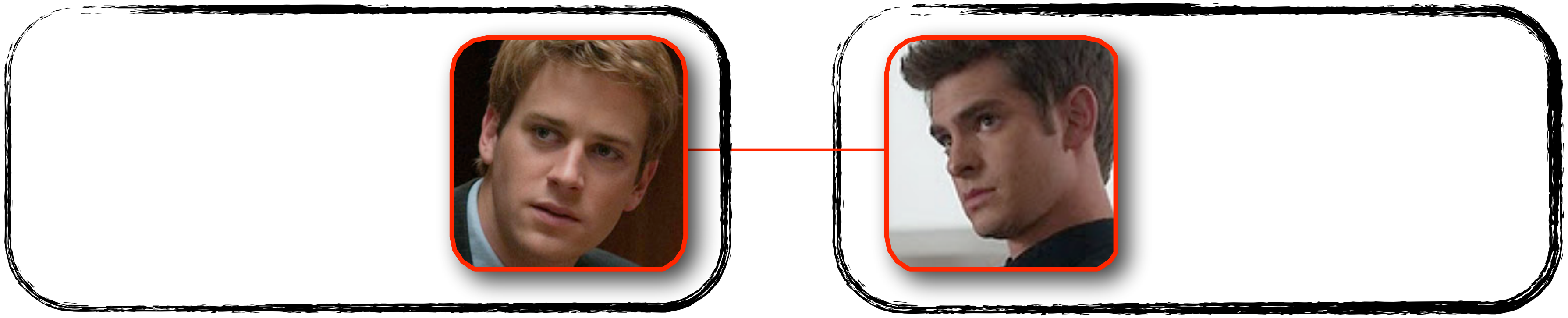

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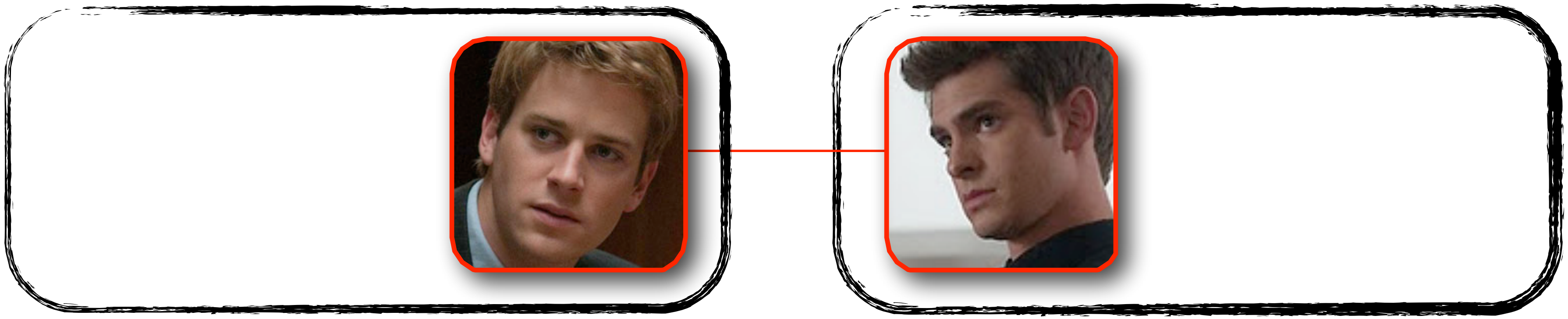

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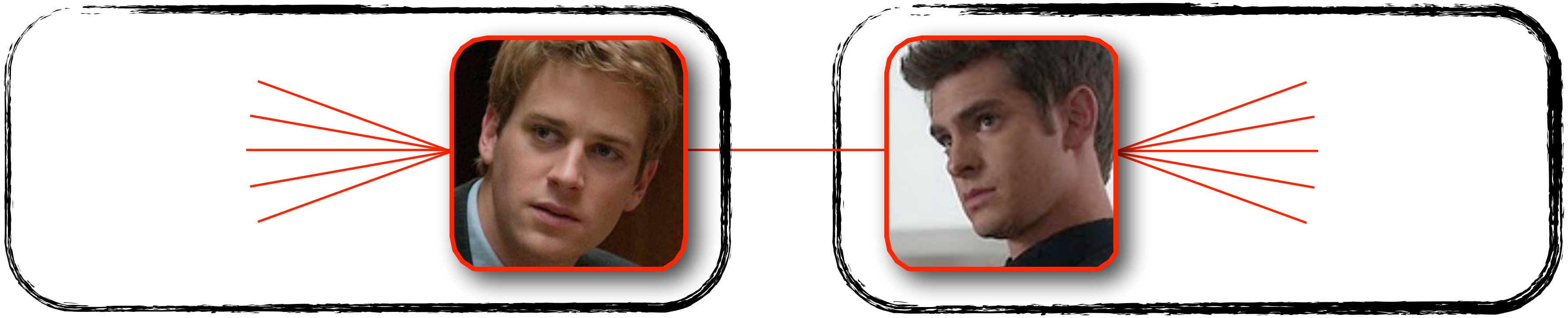

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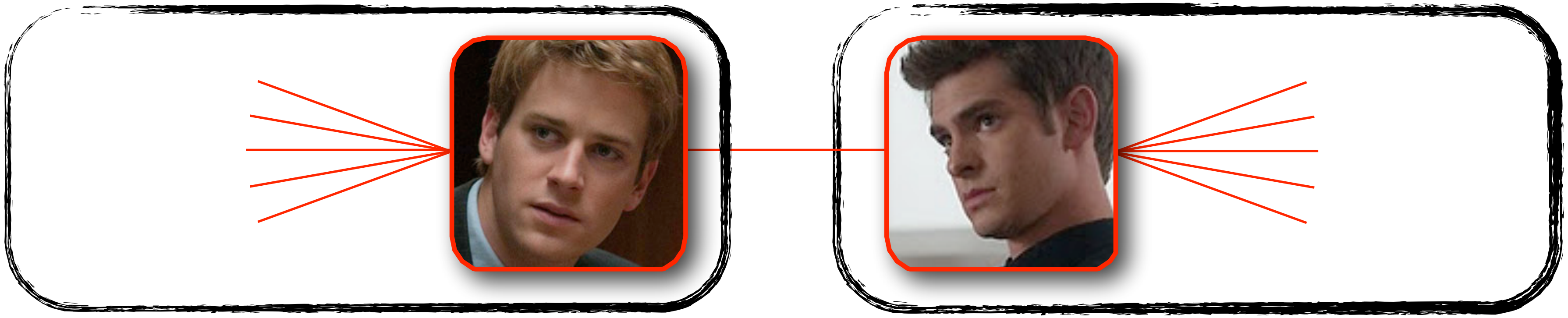

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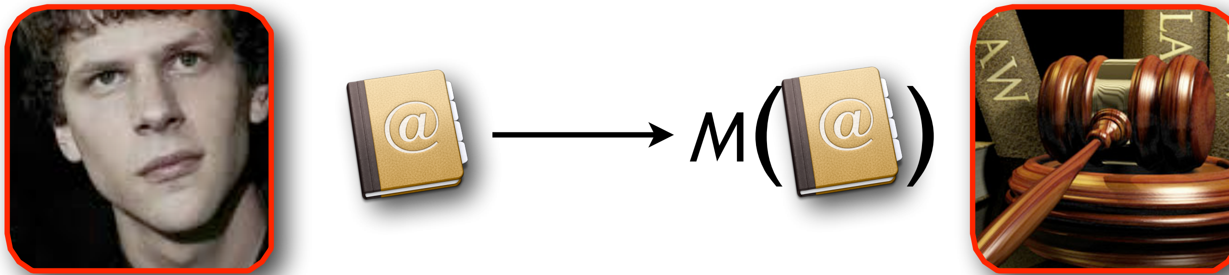

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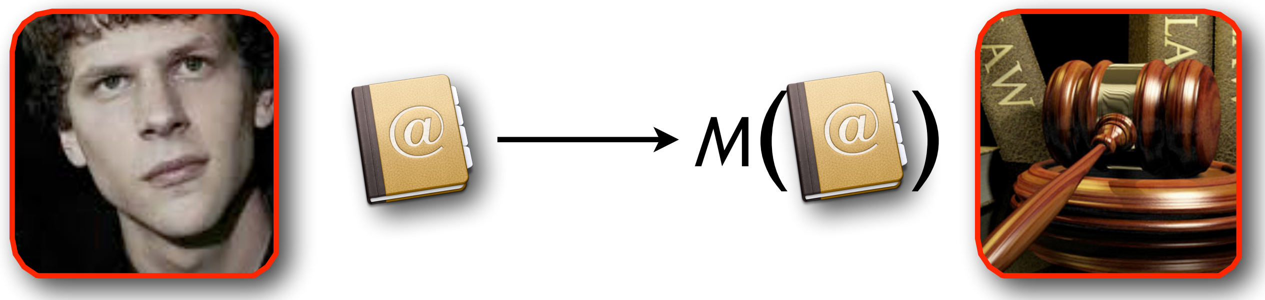

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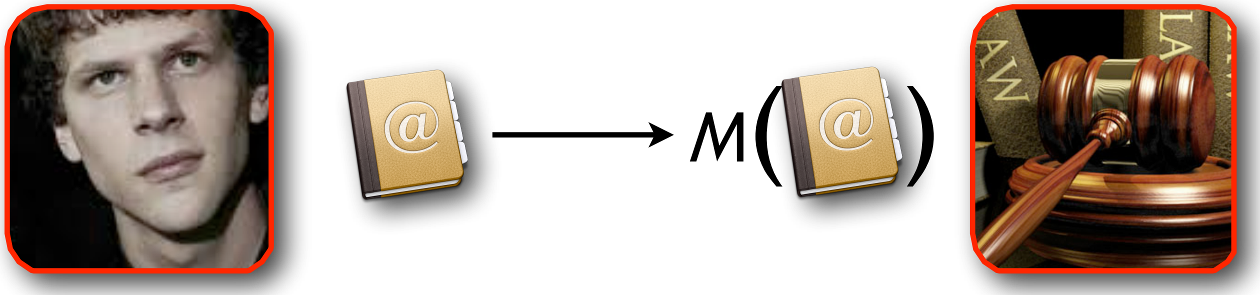

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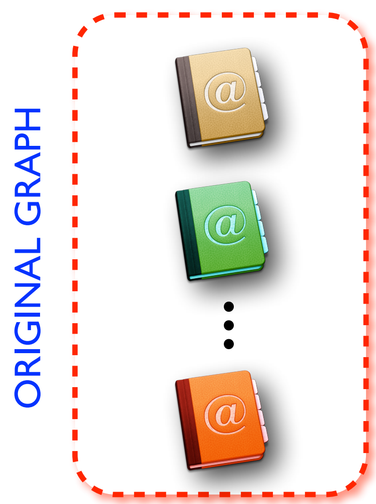

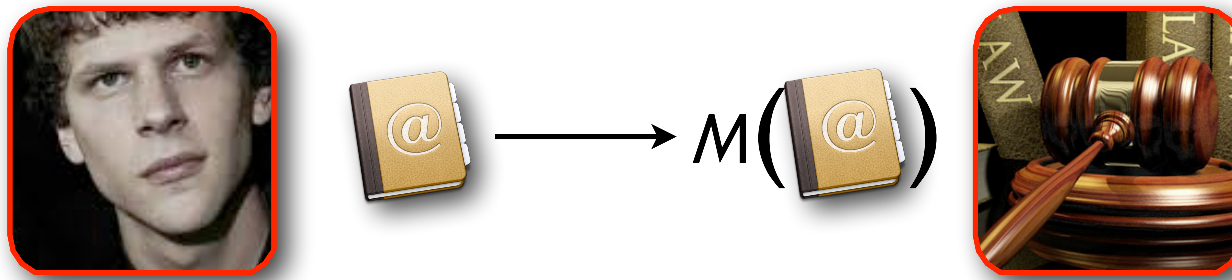

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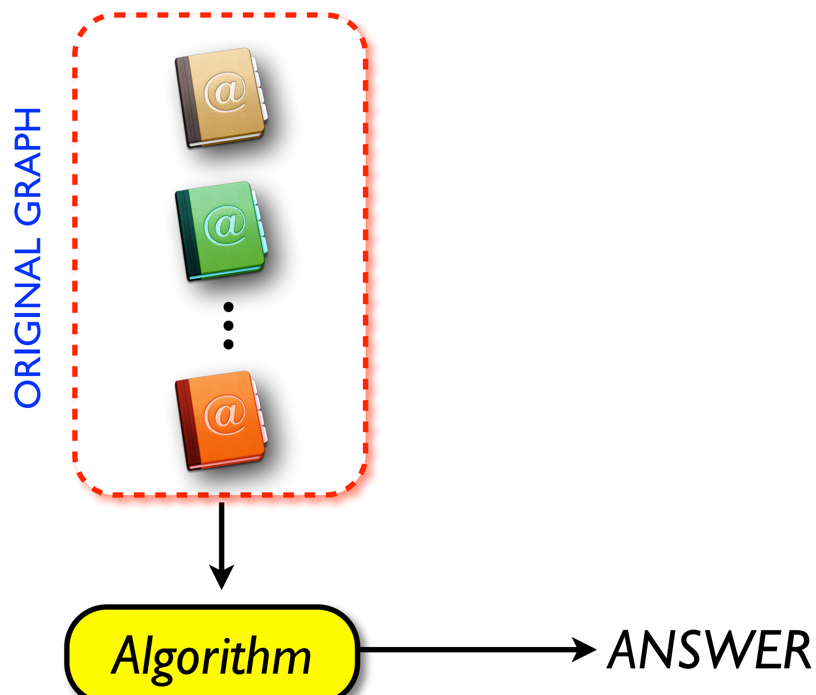

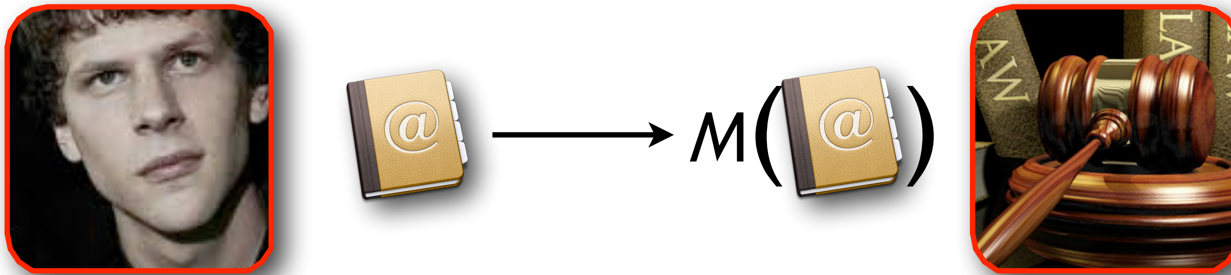

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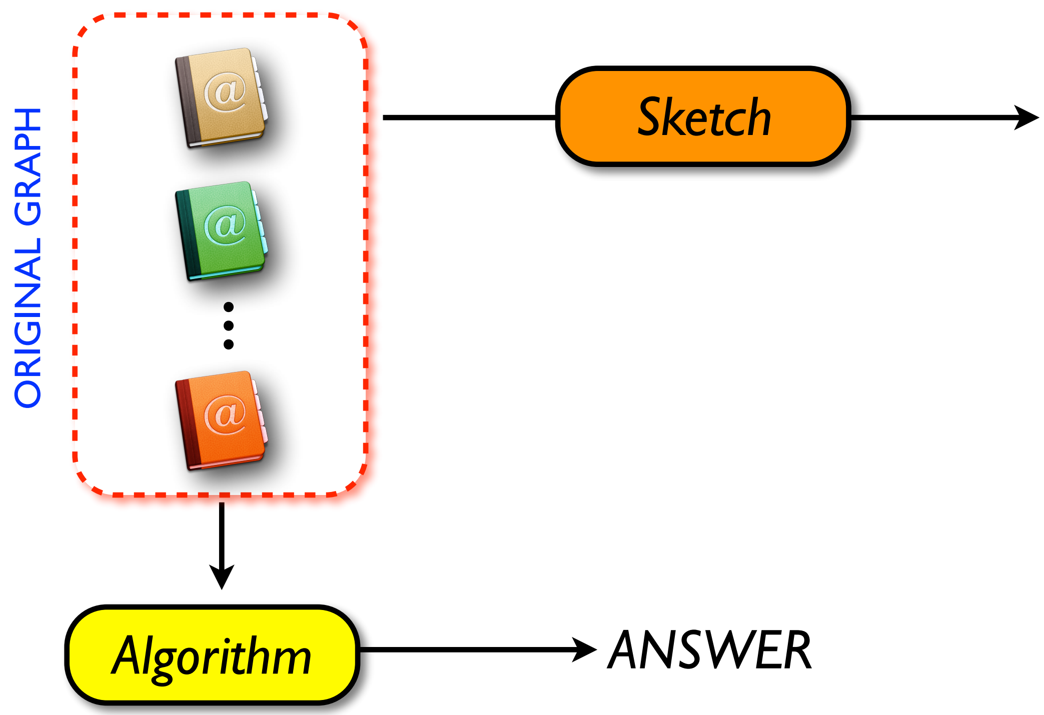

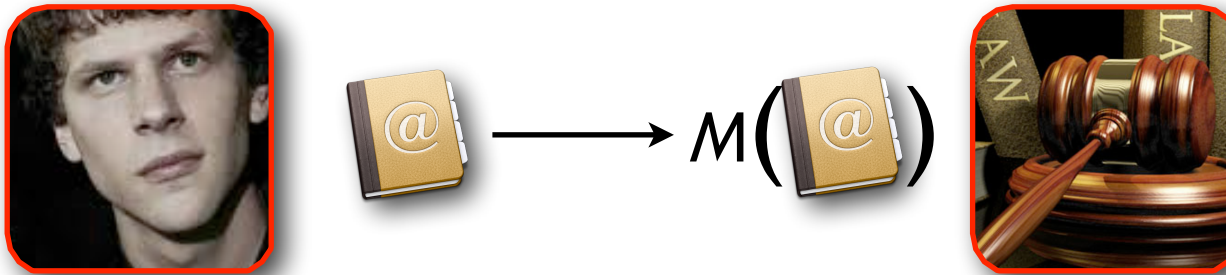

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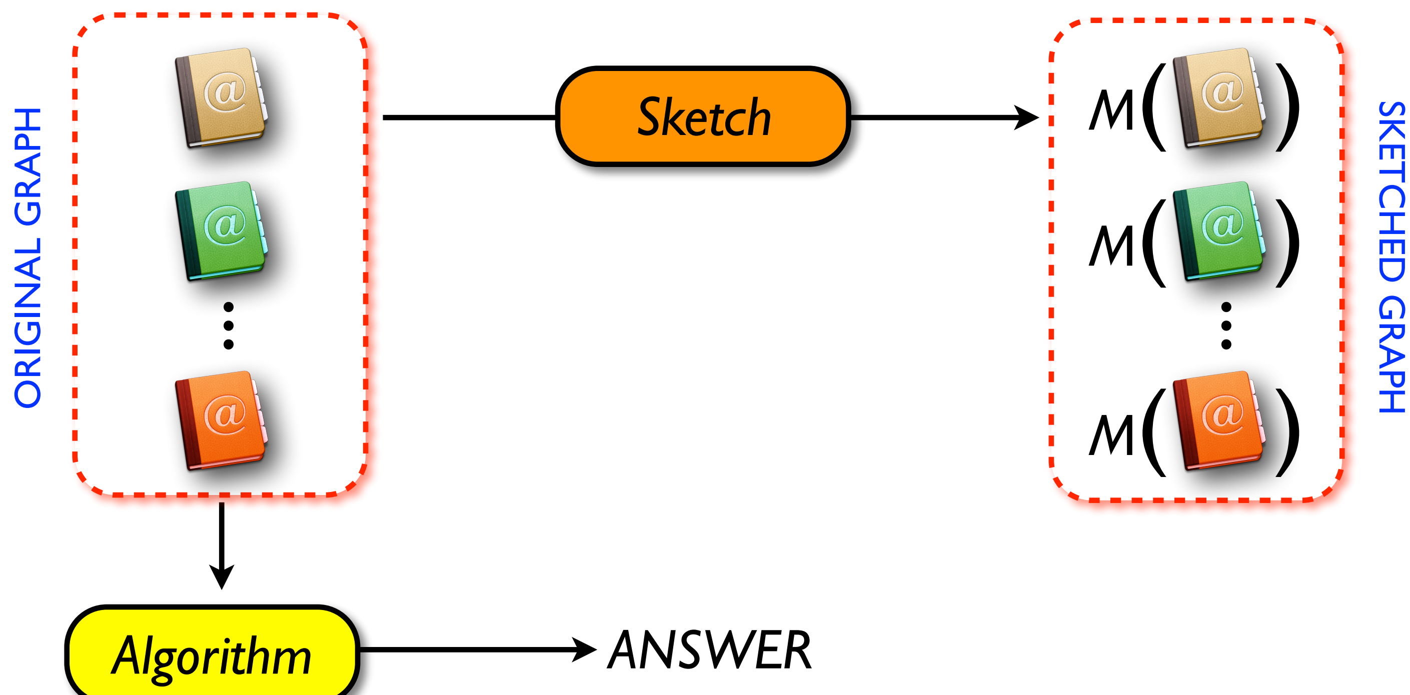

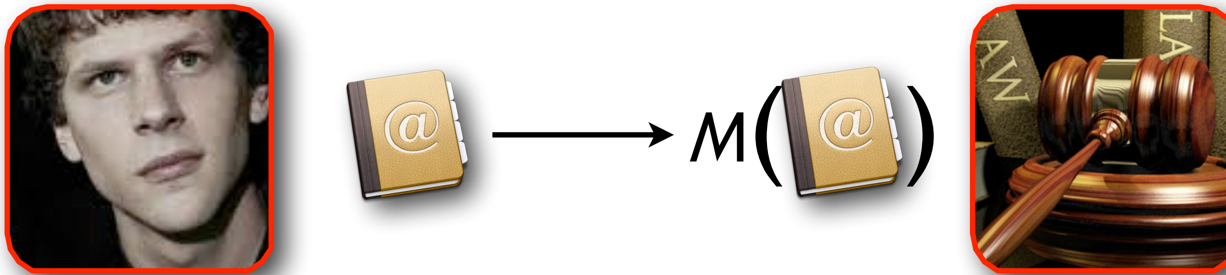

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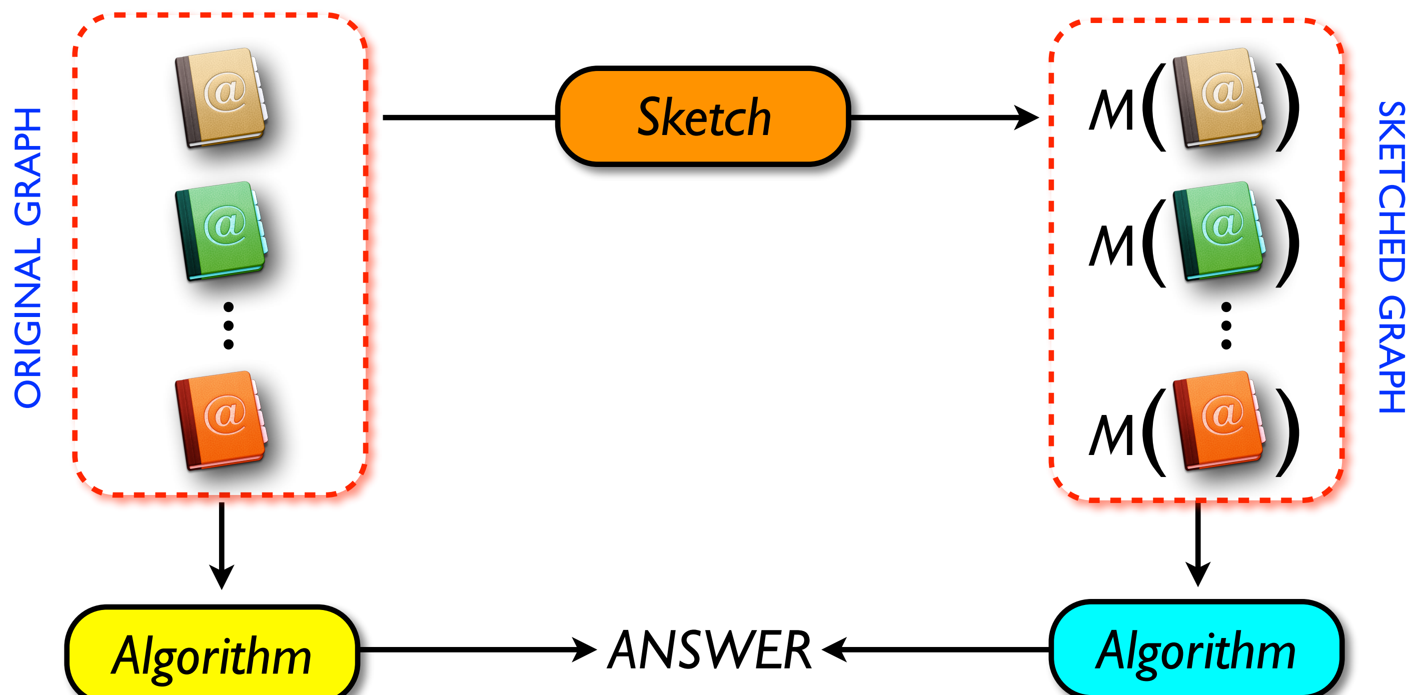

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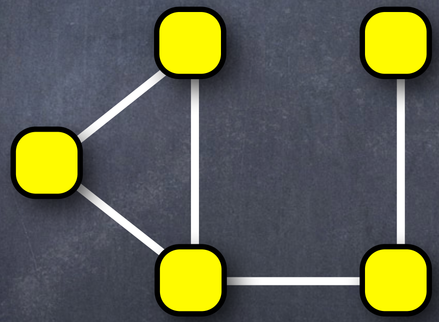

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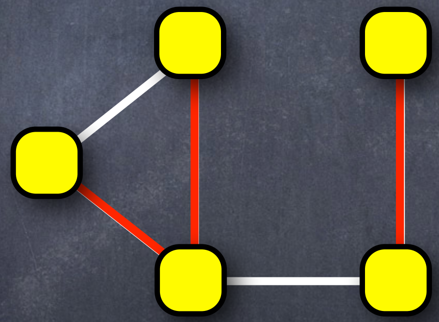

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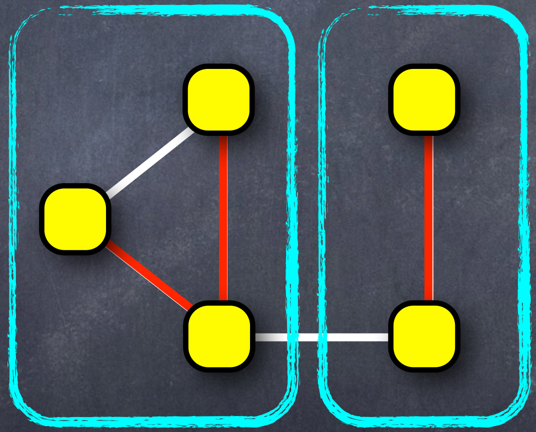

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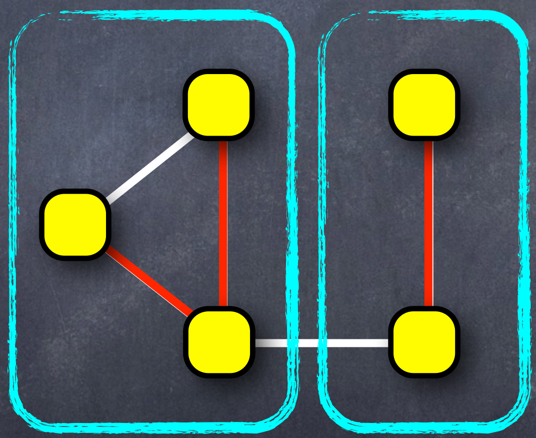

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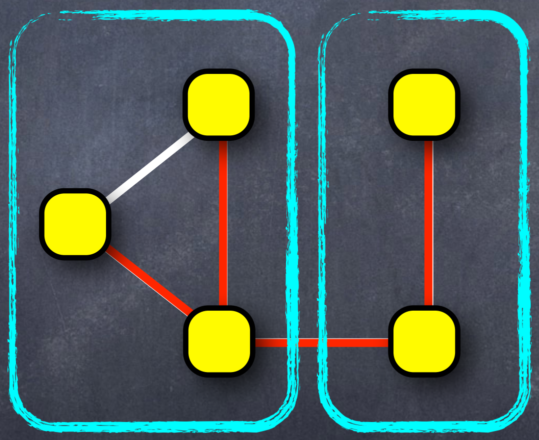

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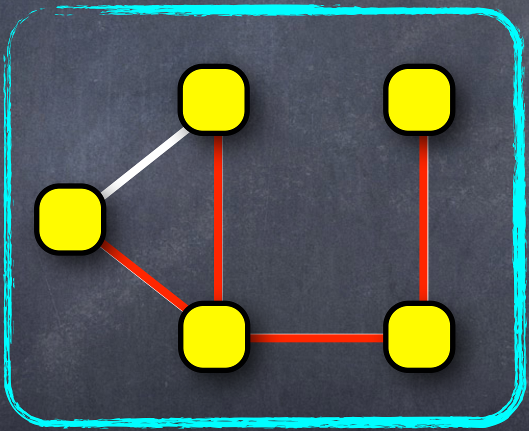

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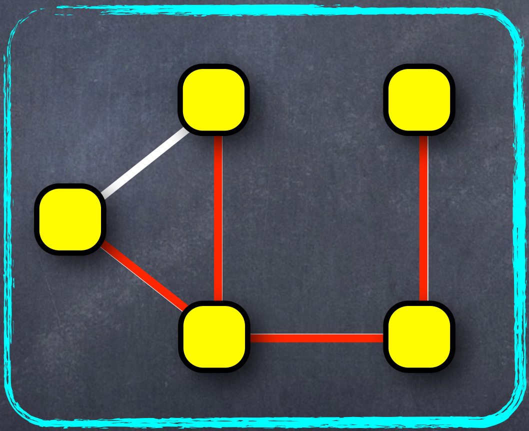

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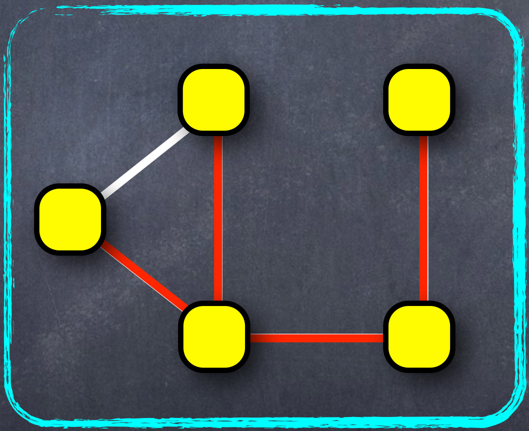
  - Repeat until no edges between connected comp.



- Lemma After O(log n) rounds selected edges include spanning forest.

Ingredient 2: Sketching Neighborhoods

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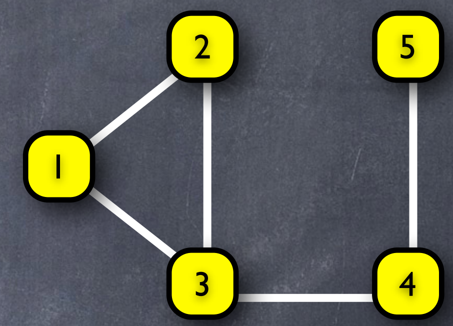
$\mathbf{a}_1 =$	1	1	0	0	0	0	0	0	0	0



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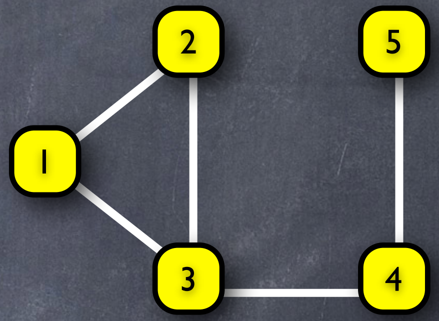

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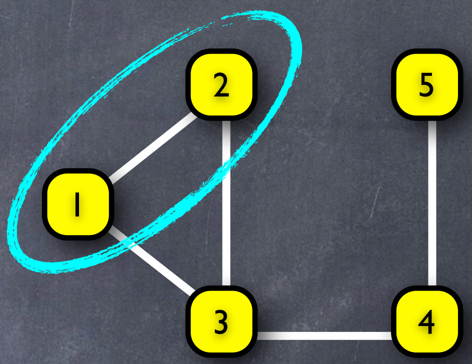

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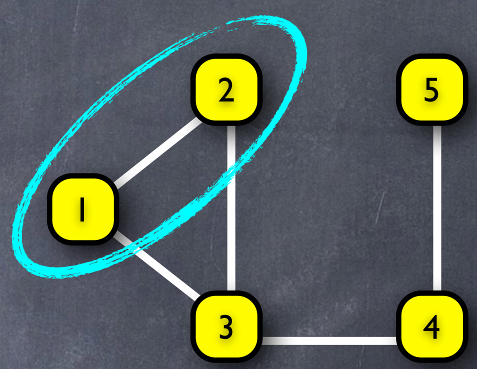

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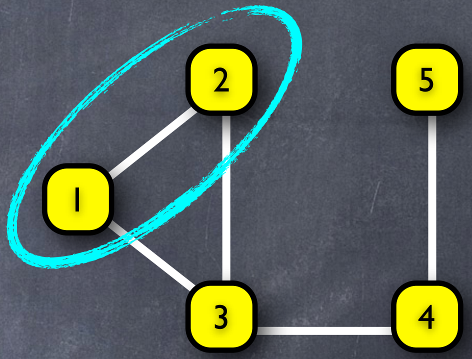

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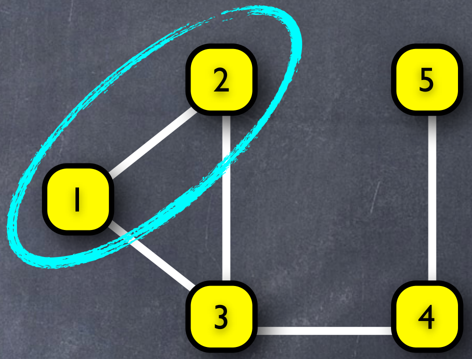

- Lemma For any subset of nodes $S \subset V$, non-zero entries of $\sum_{j \in S} a_j$ are edges across cut (S,V\S)
- Player j sends $M(a_j)$ where M is "l_0 sampling" sketch.

Recipe: Sketch & Compute on Sketches

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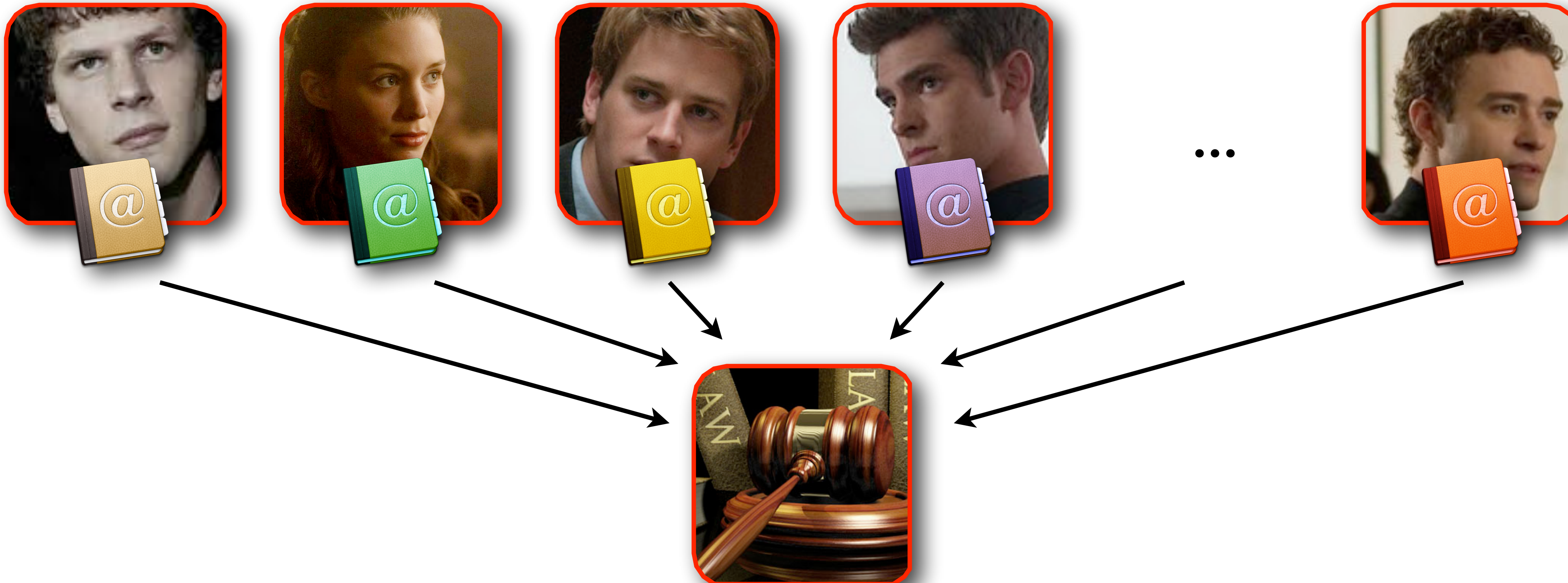
Detail: Actually each player sends log n independent sketches $M_1 a_j$, $M_2 a_j$, … and central player uses $M_i a_j$ when emulating i^{th} iteration of the algorithm.



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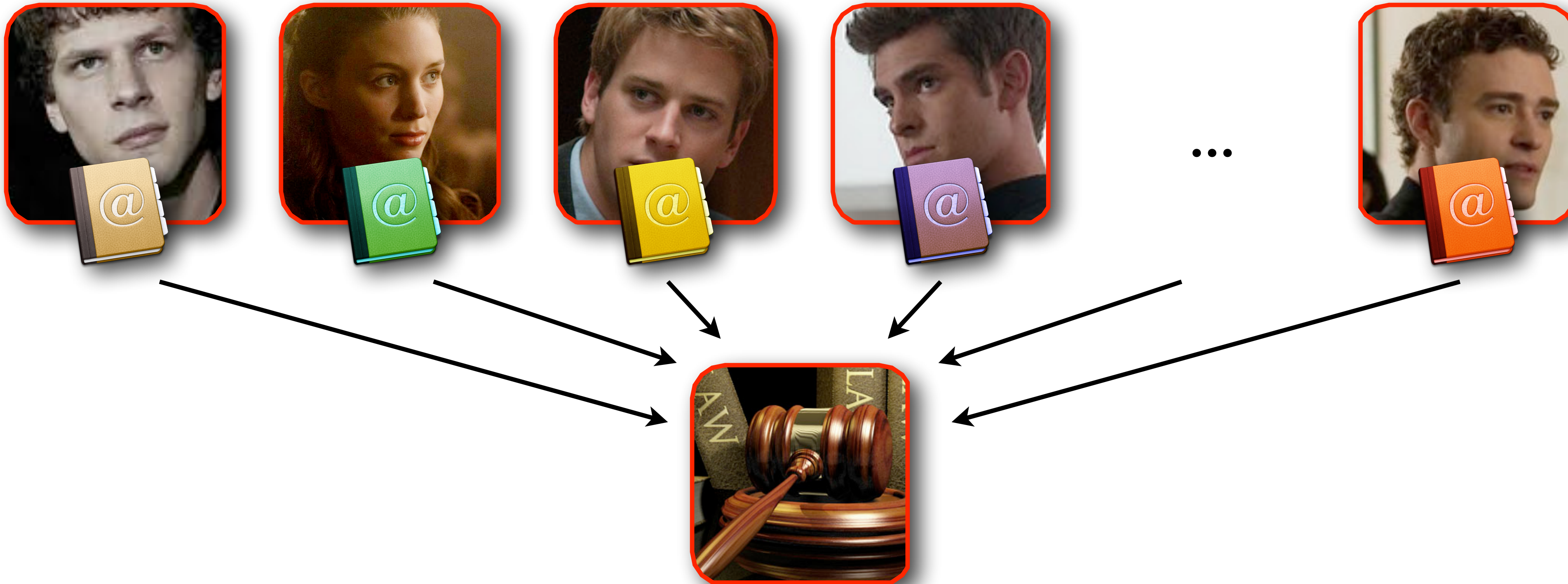

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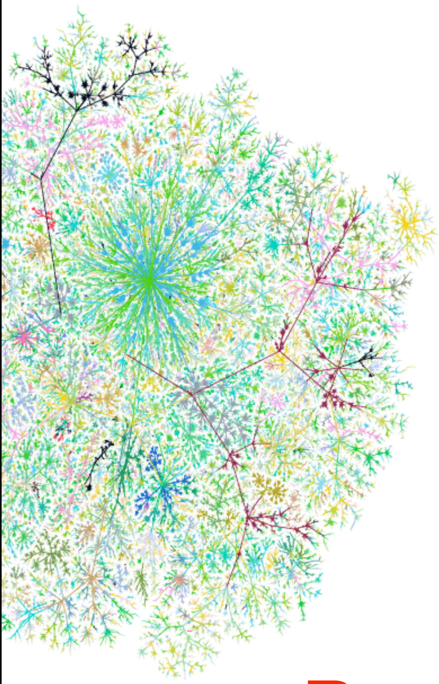
- *Various extensions* For example, with Õ(k) bit messages, can find all edges that participate in cuts of size less than k.



Part II

Sketching

What is sketching?

Surprising connectivity example

Revisiting graph cuts and sparsification

- *Thm* O(ε^{-2} polylog n) bit messages suffice for central player to construct sparsifier and approx all graph cuts.

Guha, McGregor, Tench [PODS 15], *Kapralov et al.* [STOC 14]

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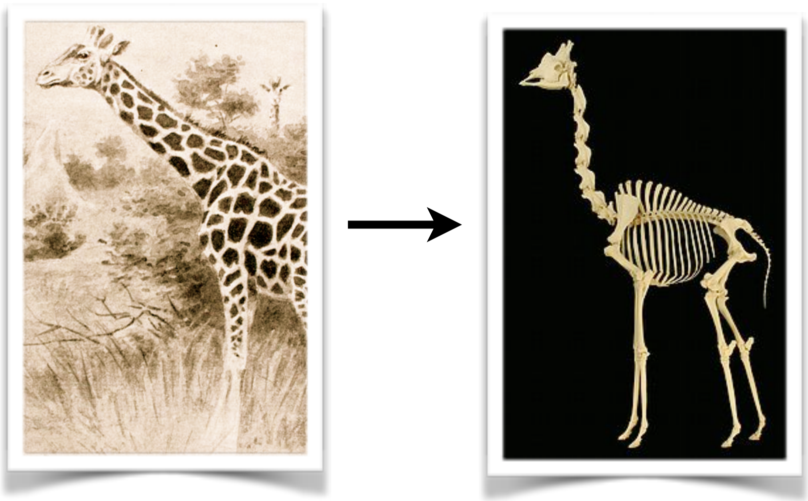

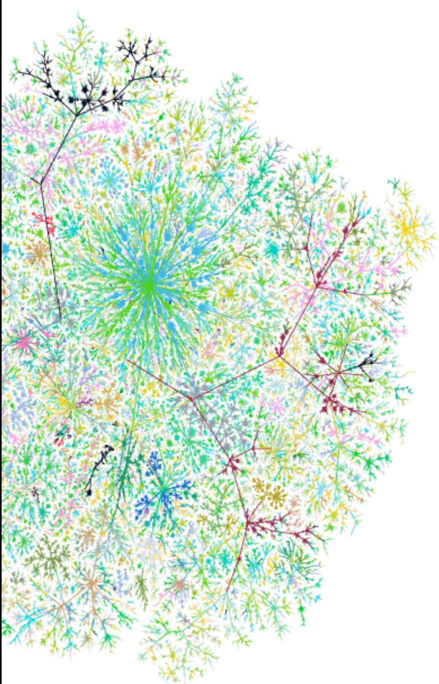

Part III

Streaming

Revisiting Matching
Correlation Clustering
Coloring Graphs
Coverage and Submodular Maximization

- *Two Main Graph Stream Models*
  - *Insert-Only Model:* Input is a stream of edges.
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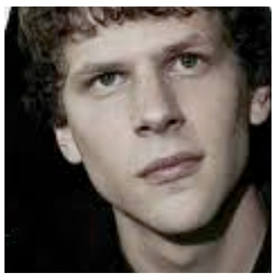

Mark and Erica are now friends.

Like · Add Friend

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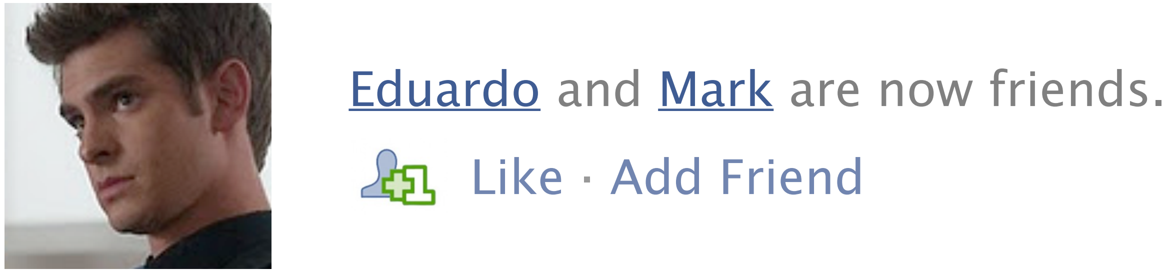

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Tyler and Cameron are friends with Mark.

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Sean and Mark are now friends.

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Eduardo and Mark are no longer friends.

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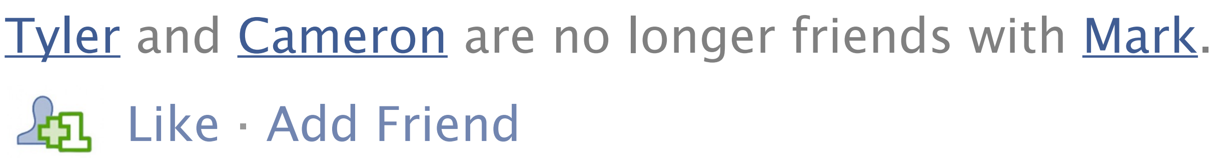

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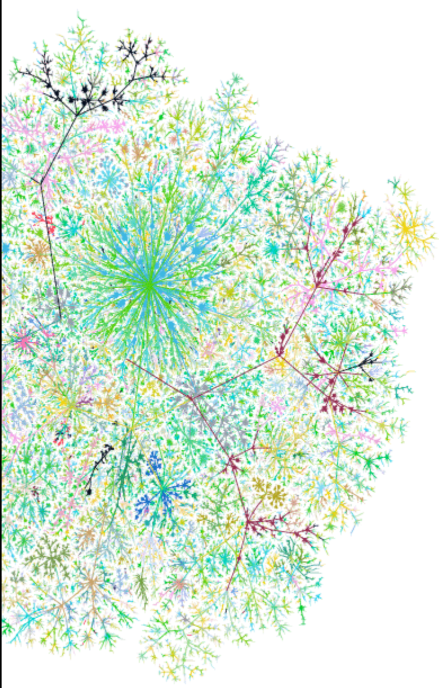

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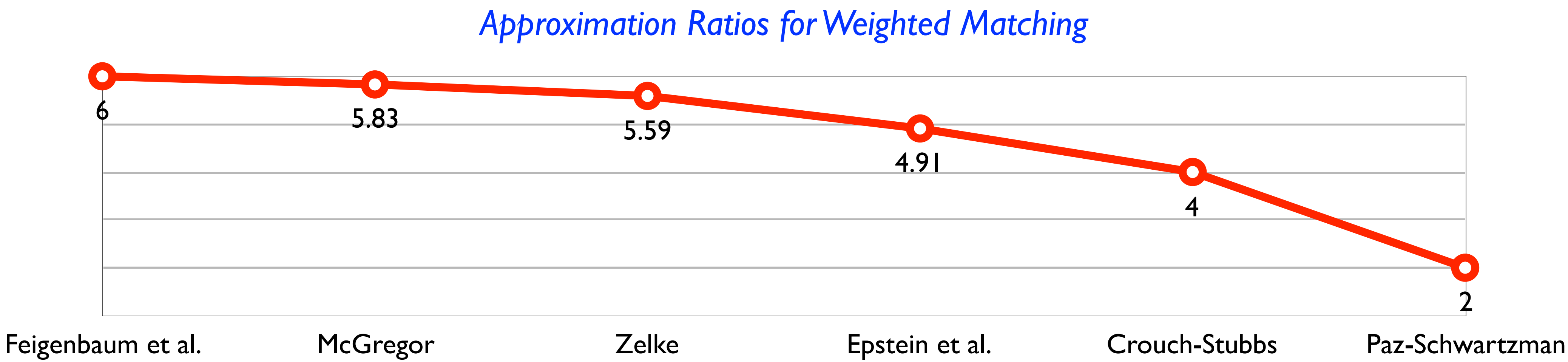

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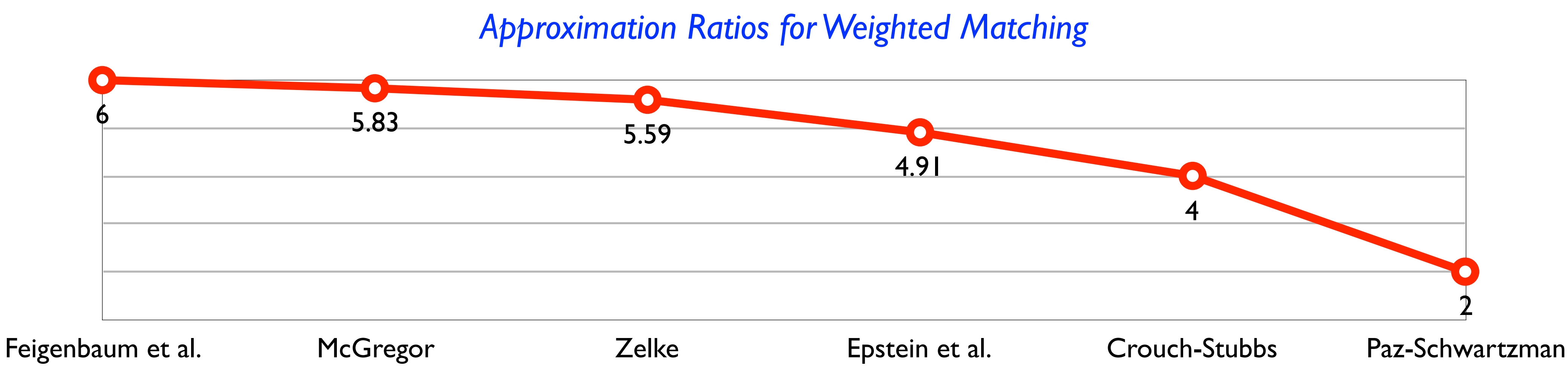

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Paz, Schwartzman [SODA 17]

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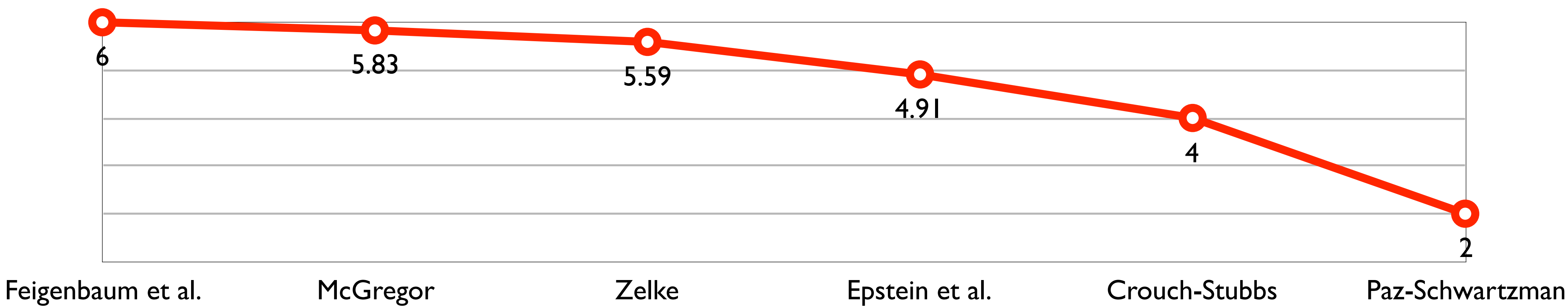

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? Improve result for sparse graphs? Graph has arboricity α if all subgraphs have average degree < α. Planar graph has α=3.

- *Thm* $\alpha+2+\varepsilon$ approx of matching size in O(polylog n) space.

Cormode et al. [ESA 17], *McGregor, Vorotnikova* [SOSA 18]

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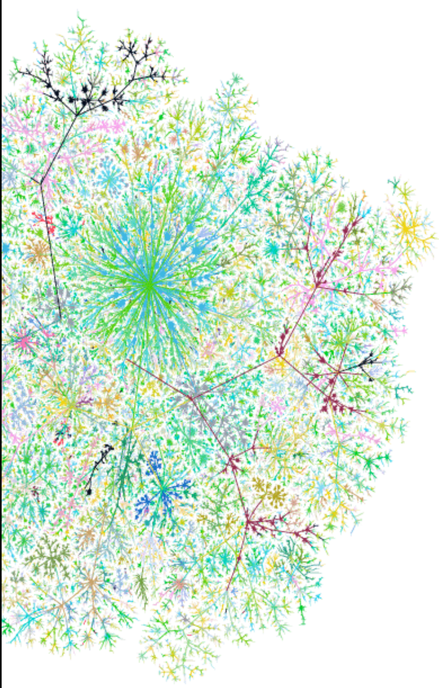
- Can show a) the current sample size is always small and b) size of final sample and g yields good approx for s.



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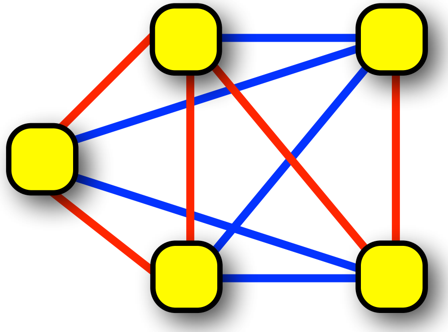

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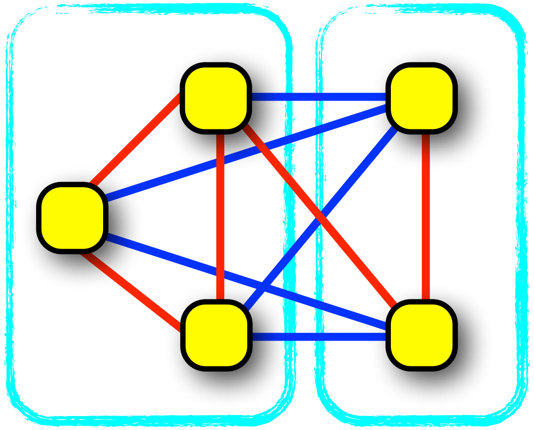

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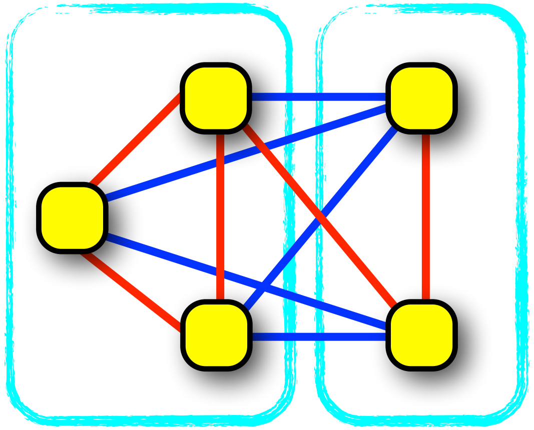
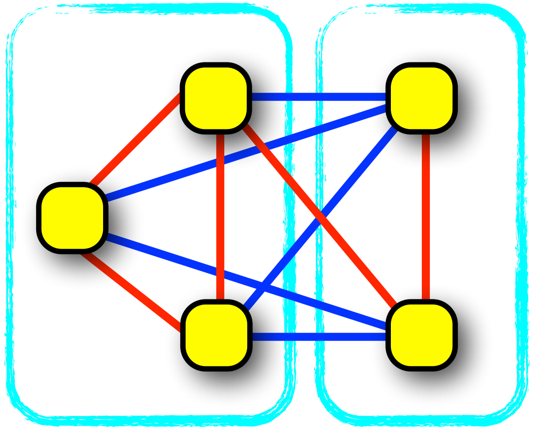


- *Correlation Clustering* Find partition minimizing # sad edges.

See tutorial *Bonchi, Garcia-Soriano, Liberty* [KDD 14]

- Consider a complete graph where edges are labelled attractive or repulsive. Given a node partition, an attractive edge is sad if it is cut and a repulsive edge is sad if it is not cut.



- *Correlation Clustering* Find partition minimizing # sad edges.

  See tutorial *Bonchi, Garcia-Soriano, Liberty* [KDD 14]

- *3-Approx Algorithm* a) Pick random node. b) Form cluster with it and its attracted neighbors. c) Remove cluster from graph and repeat until nodes remain. *Ailon, Charikar, Newman* [J.ACM 08]

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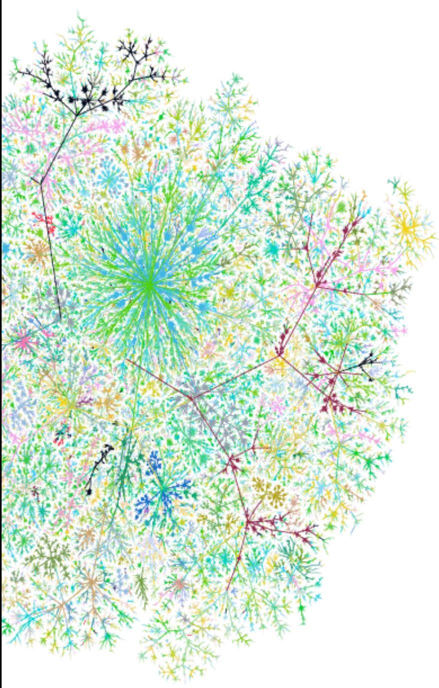
- With more work, can get $\tilde{O}(n)$ space with $O(\log \log n)$ passes. Can also find maximal independent sets.



Part III

Streaming

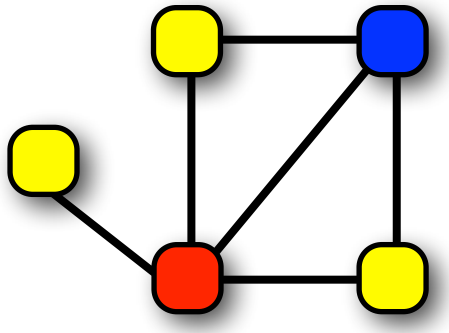
Revisiting Matching

Correlation Clustering

Coloring Graphs

Coverage and Submodular Maximization

- *Coloring* With min number of colors, assign a color to every node such that no edge has monochromatic endpoints.



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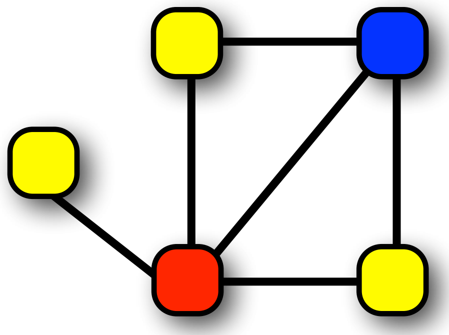

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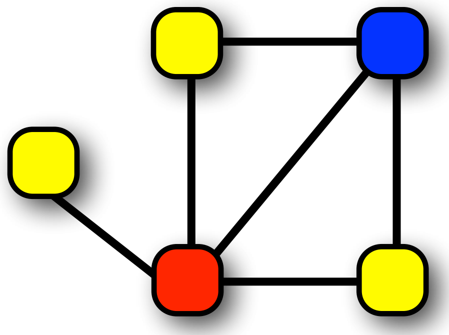

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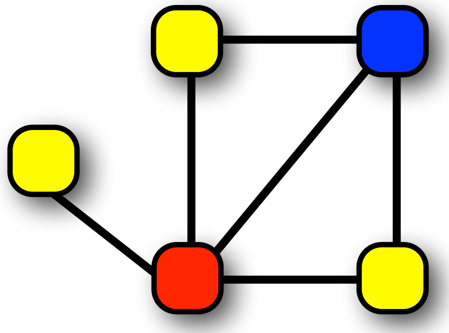

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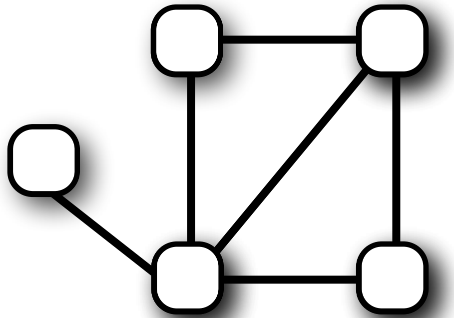
- $O(\Delta \log \log n)$ passes via independent sets. Let's do better!

- (1+ε)Δ *Coloring* a) Randomly color with Δ/r colors. b) Store edges E' with monochromatic endpoints. c) Shade colors such that E' edges no longer monochromatic. *Bera, Ghosh* [ArXiv 18]



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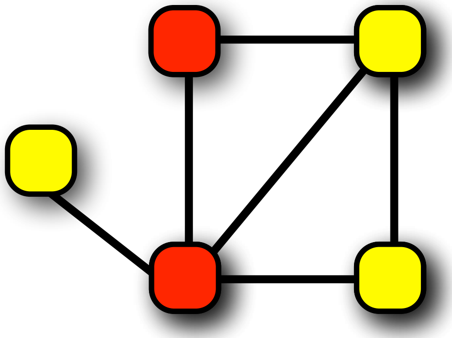

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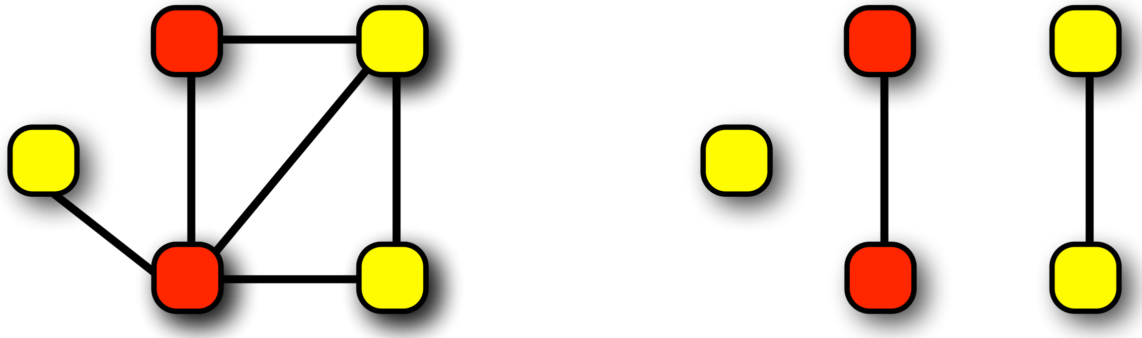

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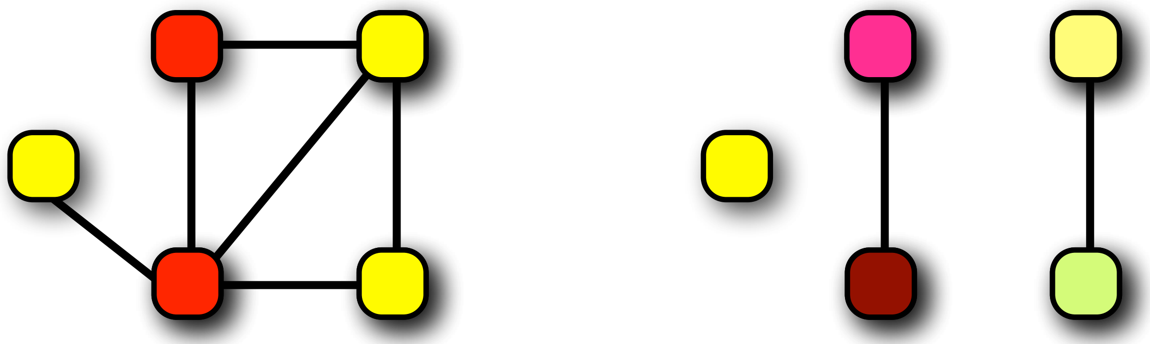

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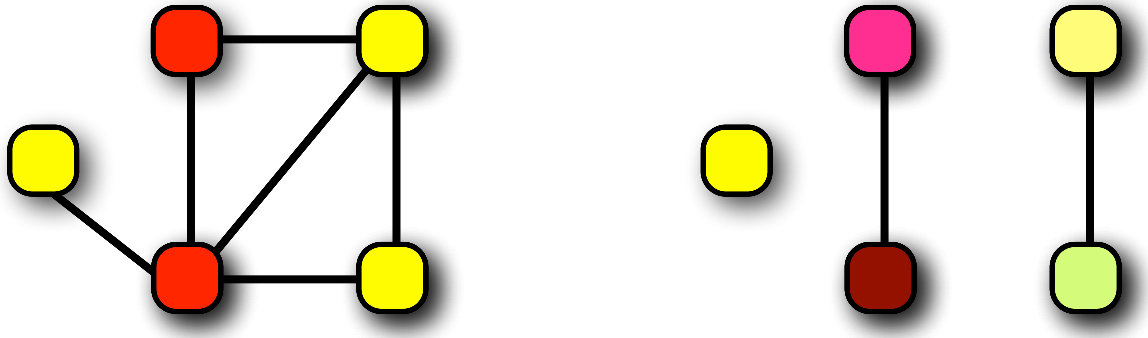

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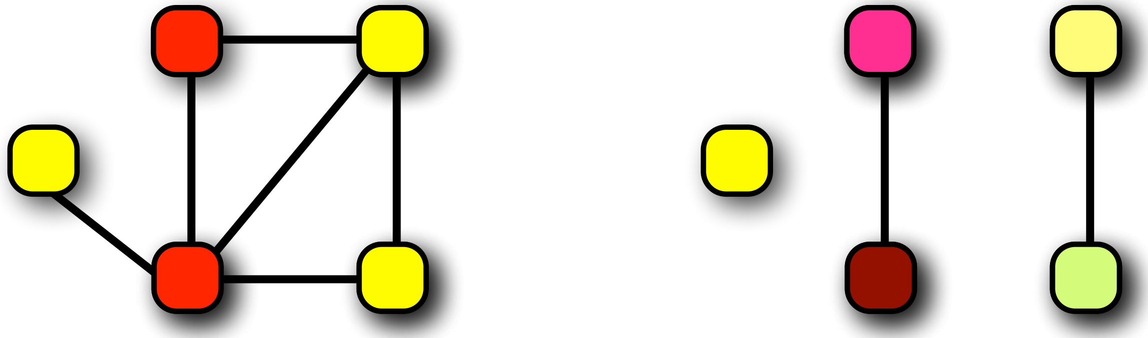

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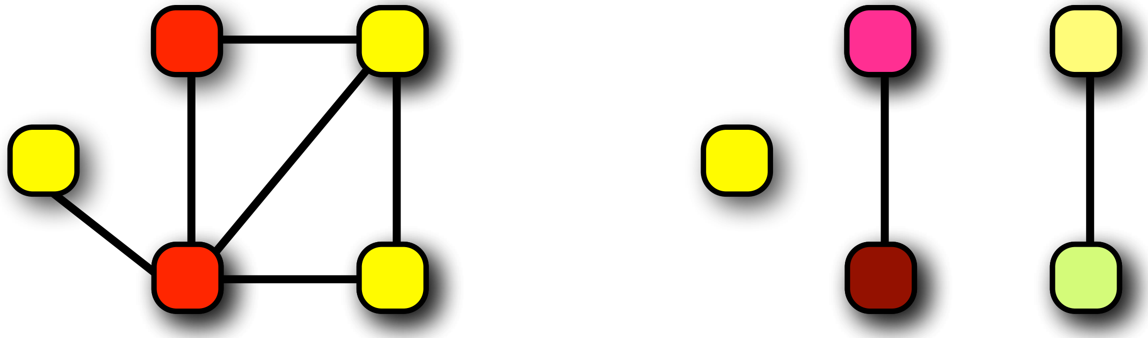

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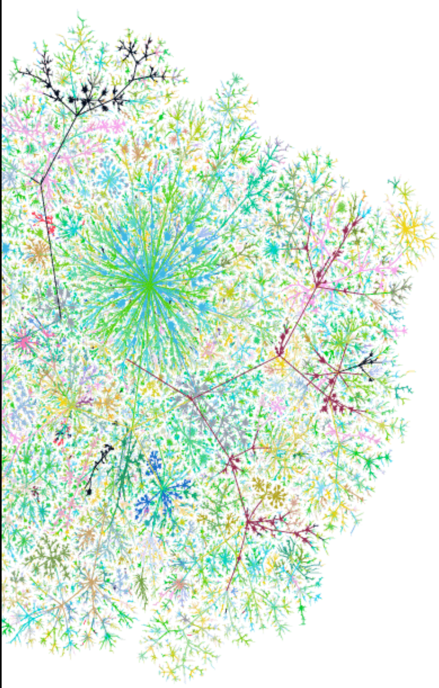
- *Δ+1 Coloring Idea* For node v, pick $S_v \subset \{1,\ldots,\Delta+1\}$ of $O(\log n)$ random colors. May assume v's color in S_v. *Assadi et al.* [ArXiv 18]



Part III

Streaming

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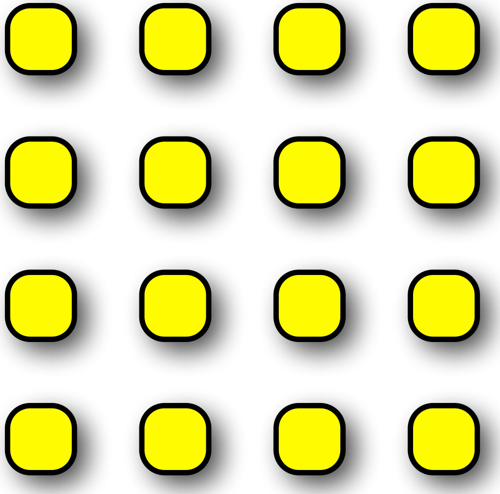
- ***Max-k-Coverage*** Given a stream of subsets $S_1, \ldots, S_m$ of $[n]$, find C that maximizes $f(C)=|\cup_{i\in C} S_i|$ subject to $|C|\leq k$.



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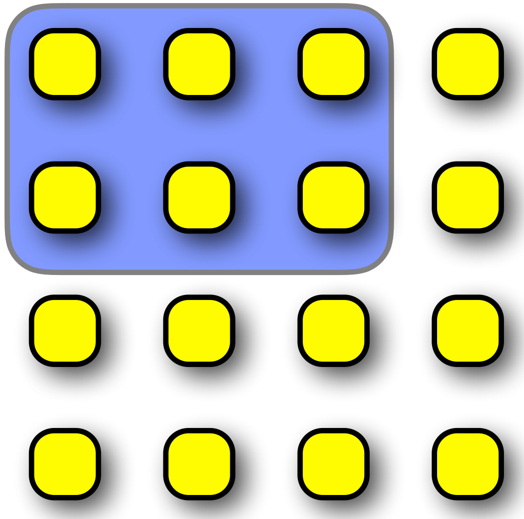

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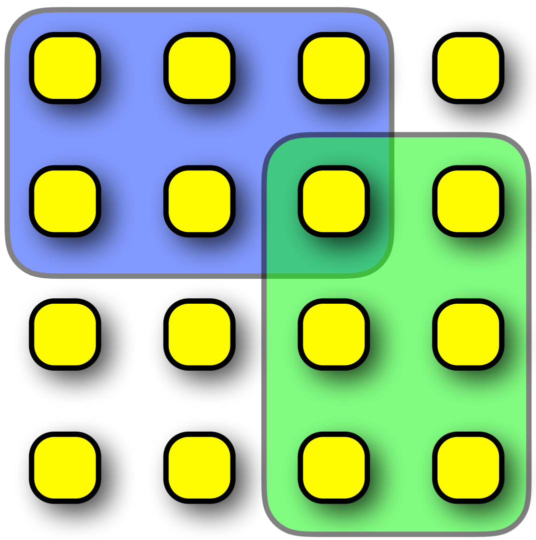

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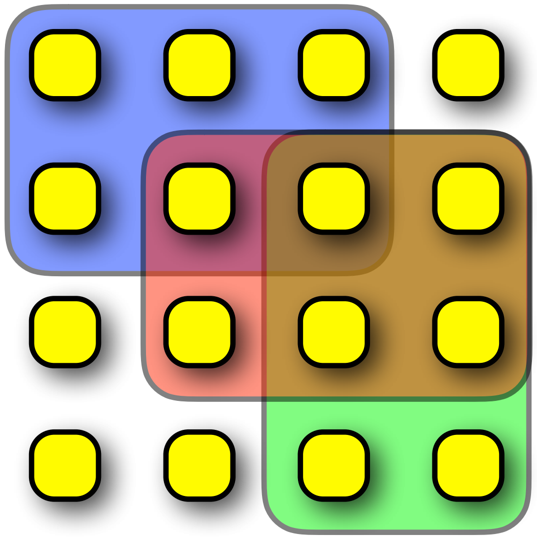

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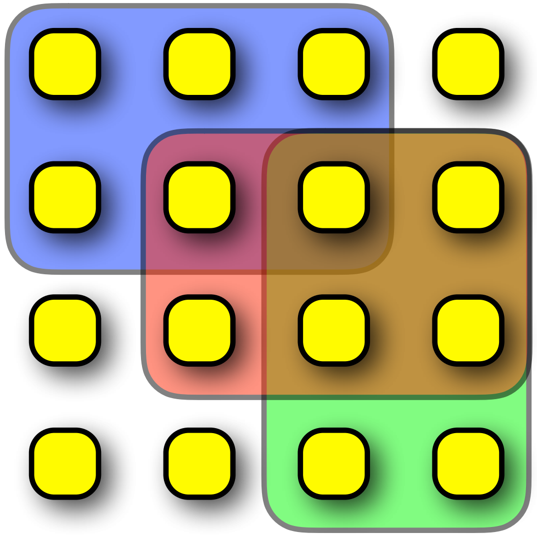

- ***Submodular Functions*** f is sub-modular if for $A \subset B$ and $x \notin B$,

$$f(A \cup \{x\}) - f(A) \geq f(B \cup \{x\}) - f(B)$$

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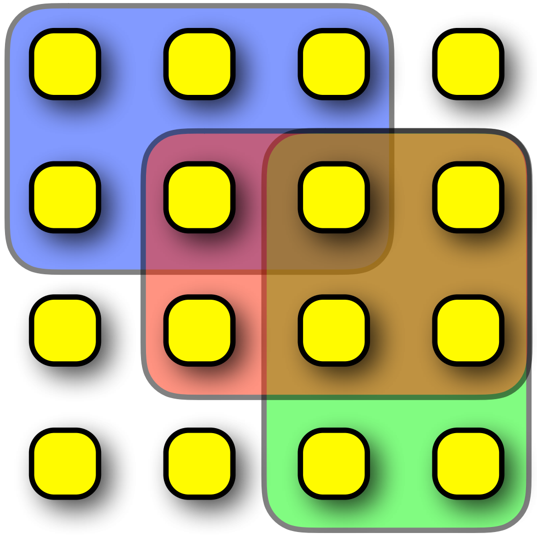

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- *Thm* $(1-\varepsilon)/2$ approx. of max-coverage in $\tilde{O}(\varepsilon^{-3}k)$ space.

McGregor, Vu [ICDT 17]

- *Algorithm* Guess g such that $OPT \leq g \leq (1+\varepsilon)OPT$. Add first $\leq k$ sets that each cover at least $g/(2k)$ new elements.

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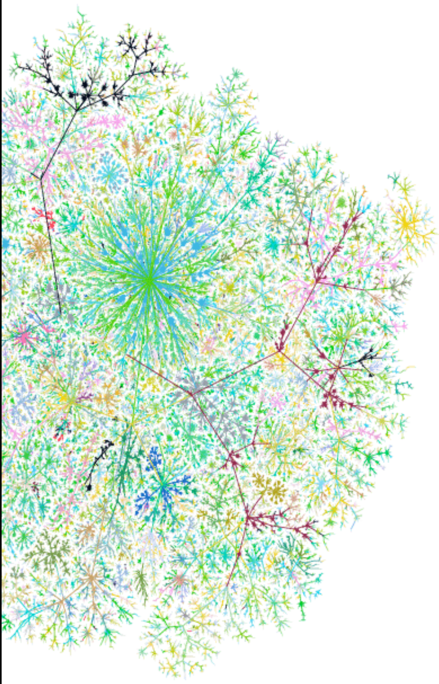
- *Generalizations* Constant passes for $\approx 1-1/e$ approx. Extends to other monotone submodular function. Other work on non-monotone functions, beyond cardinality constraints, etc.

McGregor, Vu [ICDT 17], *Bateni et al.* [SPAA 17], *Assadi* [PODS 17]



Thanks! Over to Sudipto…