Biased Normalized Cuts*

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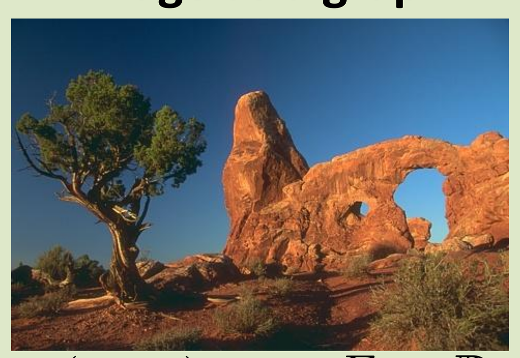
Abstract

We present a modification of "Normalized Cuts" to incorporate priors which can be used for constrained image segmentation. Compared to previous generalizations of "Normalized Cuts" which incorporate constraints, our technique has two advantages. First, we seek solutions which are sufficiently "correlated" with priors which allows us to use noisy top-down information, for example from an object detector. Second, given the spectral solution of the unconstrained problem, the solution of the constrained one can be computed in small additional time, which allows us to run the algorithm in an interactive mode. We compare our algorithm to other graph cut based algorithms and highlight the advantages.

Normalized Cuts for Image Segmentation

Image as a graph



$G = (V, E) \quad w : E \to \mathbb{R}_{\geq 0}$

w is an affinity function

Graphs and Laplacians

$$A_G \in \mathbb{R}^{V \times V}, A_G(i,j) = w(i,j)$$

$$D_G(i,i) = \sum_{j \in V} w(i,j), D_G(i,j) = 0$$

$$L_G \stackrel{\text{def}}{=} D_G - A_G$$

$$\mathcal{L}_G \stackrel{\text{def}}{=} D_G^{-1/2} L_G D_G^{-1/2}$$

Normalized Cut

$$\text{Ncut}(S, \bar{S}) \stackrel{\text{def}}{=} \frac{\text{cut}(S, \bar{S})}{\text{vol}(S)} + \frac{\text{cut}(S, \bar{S})}{\text{vol}(\bar{S})}$$

$$\text{cut}(S, \bar{S}) \stackrel{\text{def}}{=} \sum_{i \in S, j \in \bar{S}} w(i,j)$$

$$\text{vol}(S) \stackrel{\text{def}}{=} \sum_{i \in S, j \in V} w(i,j)$$

Spectral Relaxation

$$\min_{x \in \mathbb{R}^V} x^T \mathcal{L}_G x, \text{ subject to: } x^T x = 1$$

$0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n$ *eigenvalues*

$u_1, u_2, \ldots, u_n$ *eigenvectors*

The vector $v_2 = D_G^{-\frac{1}{2}} u_2$, corresponding to the second smallest eigenvalue is the solution to the relaxation.

Normalized cuts for image segmentation introduced by Shi and Malik, can be solved using spectral techniques

Bottom Up Image Segmentation

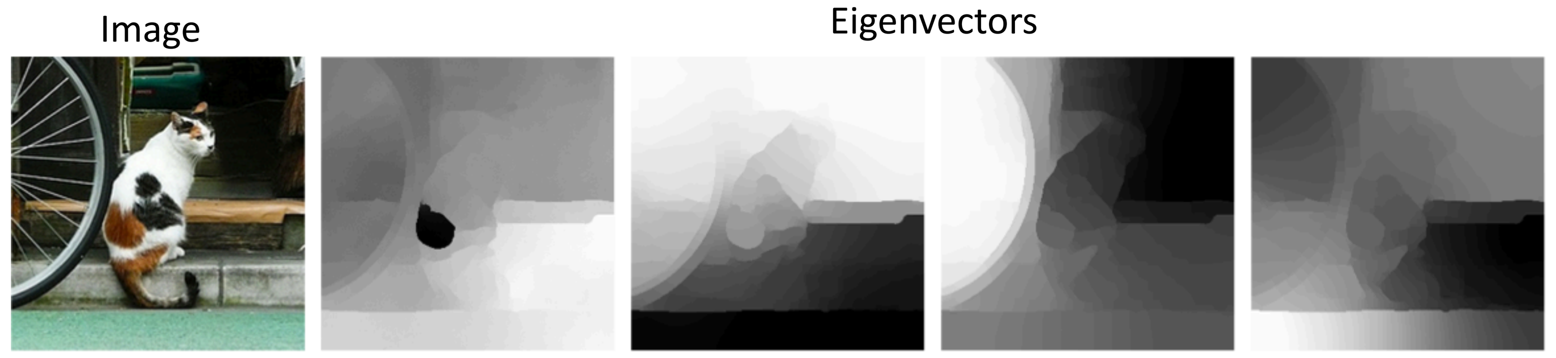


Image and its eigenvectors using the intervening contour cue with gPb (M. Maire *et.al.*, CVPR'08)
Bottom up information alone is insufficient to segment out the cat in this image.

Biased Normalized Cuts

Given a subset of vertices, $T \subseteq V$ we want cuts that minimize the "Normalized Cut" criteria, as well as are well "correlated" with *T* .

$$s_T \stackrel{\text{def}}{=} \sqrt{\frac{\text{vol}(T)\text{vol}(\bar{T})}{\text{vol}(G)}} \left(\frac{1_T}{\text{vol}(T)} - \frac{1_{\bar{T}}}{\text{vol}(\bar{T})} \right)$$

1_T Indicator vector for *T*

Define a seed vector associated with *T*

$$\text{BiasedNCut}(G, w, s_T, \kappa)$$

$$\begin{array}{ll} \text{minimize} & x^T L_G x \\ \text{s.t.} & \sum_{i,j \in V} d_i d_j (x_i - x_j)^2 = \text{vol}(G) \\ & \left(\sum_{i \in V} x_i s_T(i) d_i\right)^2 \geq \kappa \end{array}$$

Normalized Cut (first two lines) — **Bias** (last constraint)

Seek solutions that are well correlated with the seed vector.
We exploit the results in M. W. Mahoney, L. Orecchia, and N. K. Vishnoi. *A spectral algorithm for improving graph partitions. CoRR, abs/0912.0681, 2009.*

Solution using a Spectral Relaxation

Algorithm 1 Biased Normalized Cuts (G, w, s_T, γ)

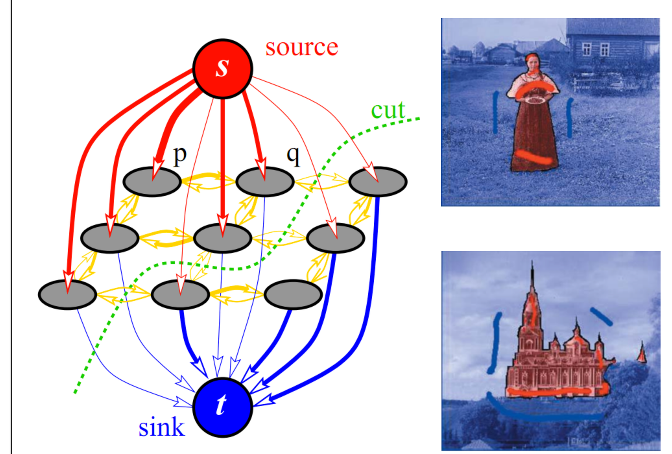
Require: Graph $G = (V, E)$, edge weight function w, seed s_T and a correlation parameter $\gamma \in (-\infty, \lambda_2(G))$

1: $A_G(i,j) \leftarrow w(i,j)$, $D_G(i,i) \leftarrow \sum_j w(i,j)$
2: $L_G \leftarrow D_G - A_G$, $\mathcal{L}_G \leftarrow D_G^{-1/2} L_G D_G^{-1/2}$
3: Compute $u_1, u_2, \ldots, u_K$ the eigenvectors of $\mathcal{L}_G$ corresponding to the K smallest eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_K$.
4: $w_i \leftarrow \frac{u_i^T D_G s_T}{\lambda_i - \gamma}$, for $i = 2, \ldots, K$
5: Obtain the biased normalized cut, $x* \propto \sum_{i=2}^{K} w_i u_i$

1. The solution is a weighted combination of the eigenvectors. The weights of the eigenvectors are proportional to the correlation with the seed vector, i.e. eigenvectors that are well correlated get up-weighted.
2. Steps 1-3, are the steps for solving Normalized Cuts.
3. In an interactive setting, only Steps 4-5 need to be repeated.
4. On natural images, eigenvalues grow quickly, so using top ***k*** eigenvectors are enough for a good approximation. We set ***k***=25 in our experiments.
5. Matrix L, D are sparse so, complexity is linear in the number of pixels.

Graph based Image Segmentation with Constraints

Min *s-t* Cuts



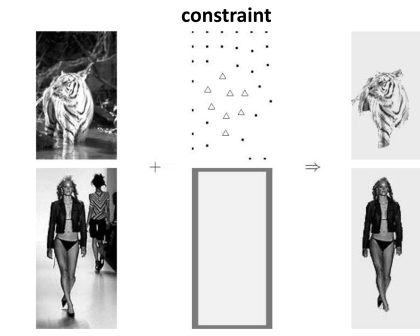
Example segmentations

1. Red/blue edges encode the prior.
2. Can be solved using max-flow/min-cut techniques. Efficient algorithms exist for grid graphs (Boykov *et.al.*)
3. Popular for interactive image segmentation, e.g. GrabCut

Constrained Ncuts (Yu & Shi)

$$\min_x \text{NCut}(G, w, x)$$

$Cx = 0$ **Encodes the prior**



$$Wx = \lambda x$$

1. Can be solved using a modified eigenvalue problem. However, this is impractical for large problems as the matrix W is dense.
2. It is hard to incorporate constraints in a soft manner.
3. Not robust to outliers.

Biased NCuts

$$\min_x \text{NCut}(G, w, x)$$

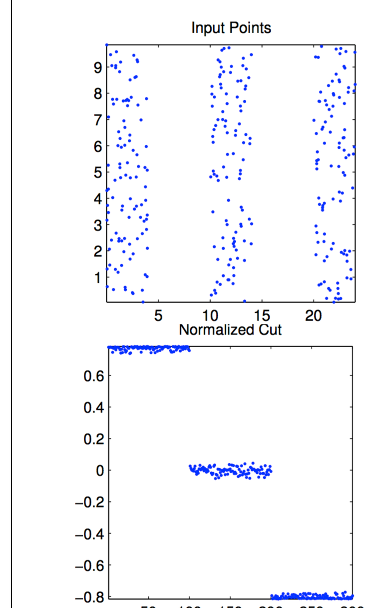
$$\left(\sum_{i \in V} x_i s_T(i) d_i\right)^2 \geq \kappa$$

Encodes the prior

1. Efficient solution using a weighted combination of eigenvectors.
2. Can be done interactively. Eigenvectors need to computed only once per graph
3. Constraints enforced in a soft manner, making it robust to outliers.
4. Can use real valued priors as seed vectors.
5. Practical for large problems.

Comparison of Biased and Constrained Normalized Cuts

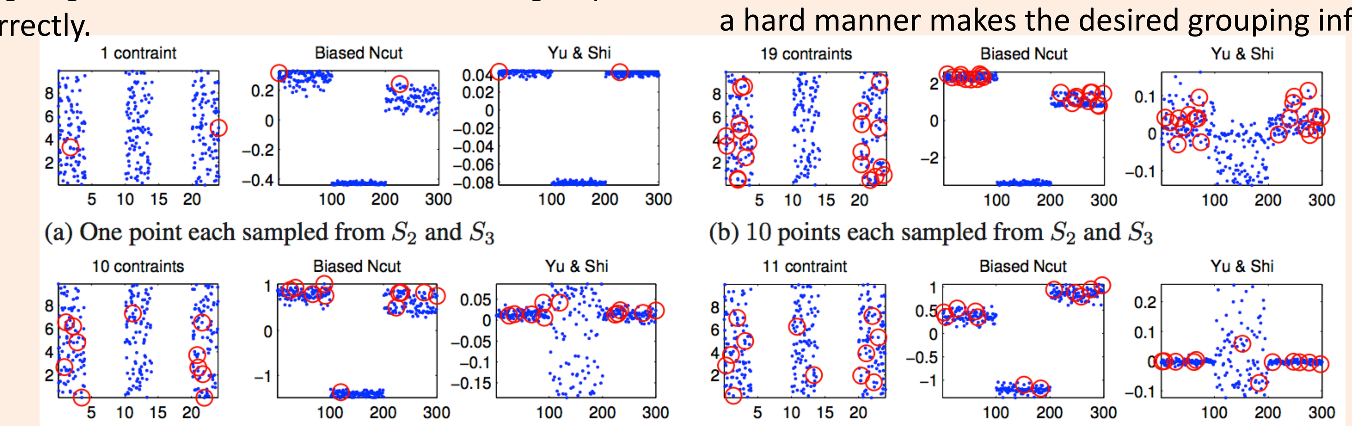
Effect of outliers and number of constraints



Grouping using Normalized Cuts without constraints

Biasing the NCut solution using the constraint that the circled points belong together. Both CNcut and BNcut group the point sets correctly.

Adding more constraints improve the BNcut solution but make the CNcut solution worse. Enforcing the constraints in a hard manner makes the desired grouping infeasible.

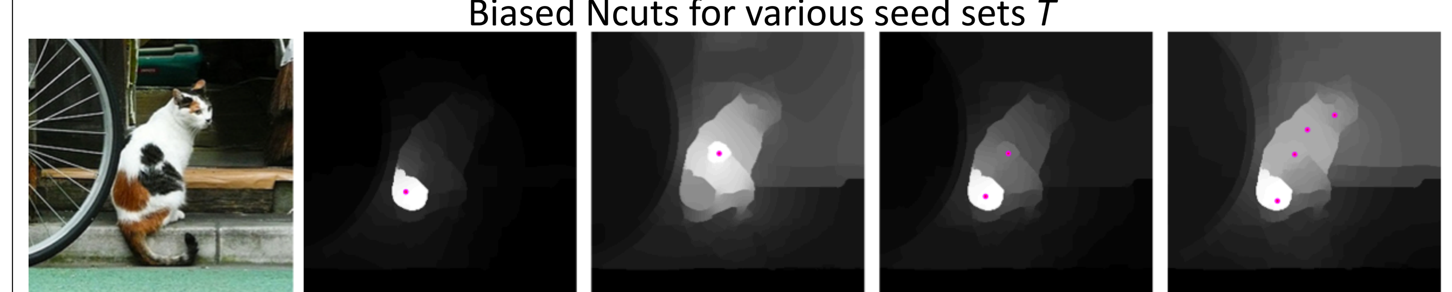


(a) One point each sampled from S_2 and S_3
(b) 10 points each sampled from S_2 and S_3
(c) 5 points each from sets S_1 and S_3 with 1 outlier from S_2
(d) 5 points each from sets S_1 and S_3 with 2 outliers from S_2

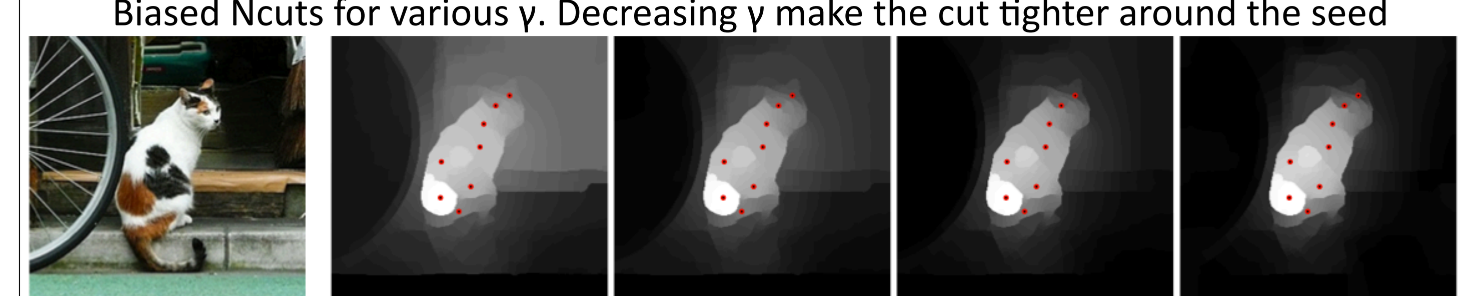
BNCut is robust when one or two outliers are added to the constraints.

Effect of Bias and Correlation Parameter (γ)

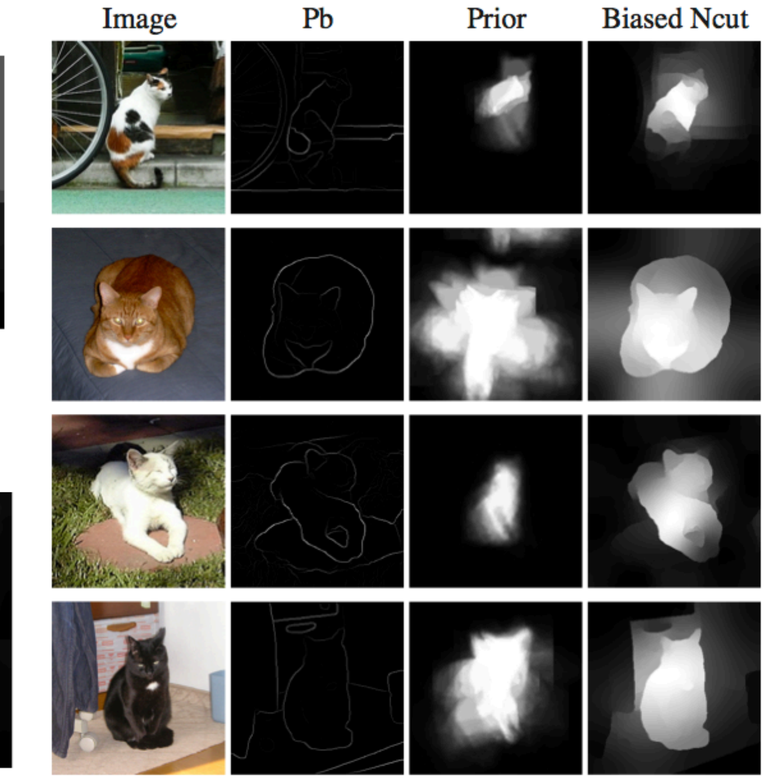
Biased Ncuts for various seed sets *T*



Biased Ncuts for various γ. Decreasing γ make the cut tighter around the seed



Top Down & Bottom Up Segmentation



Seed vectors using an object detector

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