# COMPSCI 311: Introduction to Algorithms

## Lecture 8: Greedy Algorithms – Exchange Arguments

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# Algorithm Design—Greedy

Greedy: make a single "greedy" choice at a time, don't look back.

| | Greedy |
|---|---|
| Formulate problem | |
| Design algorithm | |
| Prove correctness | ✓ |
| Analyze running time | |
| Specific algorithms | Dijkstra, MST |

Focus is on proof techniques

- Last time: "greedy stays ahead" (inductive proof)
- This time: exchange argument

# Scheduling to Minimize Lateness

- You have a very busy month: $n$ assignments are due, with different deadlines

```
Assignments:
    1: |---|           (len=1, due=2)
    2: |---o---|       (len=2, due=5)
    3: |---o---o---|   (len=3, due=6)
    4: |---o---|       (len=2, due=7)

Deadlines:
            d1           d2  d3  d4
    |---|---|---|---|---|---|---|---|---|
    0   1   2   3   4   5   6   7   8   9
```

- How should you schedule your time to "minimize lateness"?

# Scheduling to Minimize Lateness

Let's formalize the problem. The input is:

- $t_j$ = length (in days) to complete assignment $j$ (or "job" $j$)
- $d_j$ = deadline for assignment $j$

What does a schedule look like?

- $s_j$ = start time for assignment $j$ (selected by algorithm)
- $f_j = s_j + t_j$ finish time

How to evaluate a schedule?

- Lateness of assignment $j$ is $\ell_j = \begin{cases} 0 & \text{if } f_j \leq d_j \\ f_j - d_j & \text{if } f_j > d_j \end{cases}$
- Maximum lateness $L = \max_j \ell_j$

**Goal**: schedule so maximum lateness is as small as possible

# Clicker

True or false: an algorithm to minimize maximum lateness will also find a schedule that is not late, if one exists.

A. True

B. False, because the lateness function is not linear

C. False, because it minimizes the maximum lateness, whereas we want all jobs to have lateness zero

# Possible Greedy Approaches

- **Note**: scheduling work back-to-back (no idle time) can't hurt
  $\Rightarrow$ schedule determined just by order of assignments

```
1: |---|           (len=1, due=2)
2: |---o---|       (len=2, due=5)
3: |---o---o---|   (len=3, due=6)
4: |---o---|       (len=2, due=7)
```

- What order should we choose?
  - *Shortest Length*: ascending order of $t_j$.
  - *Smallest Slack*: ascending order of $d_j - t_j$.
  - *Earliest Deadline*: ascending order of $d_j$.

# Clicker

Suppose we have two jobs with lengths 1 and 10. Which idea below can we use to set deadlines to "break" smallest slack time so it does not find an optimal ordering?

A. Set the deadlines so both jobs have slack zero.
B. Set the deadlines so the short job has a little slack, and the long job has none.
C. Set the deadlines so the long job has a little slack, and the short job has none.

# Proposed Algorithm

So far, only **earliest deadline first** is optimal in all the examples we've tried.

Next, we'll prove that it's always optimal.

# Clicker

If two jobs have the same deadline, the earliest deadline first algorithm should schedule:

A. The shortest job first, because that has a higher chance of finishing before the deadline

B. The longest job first, because then its lateness will be minimized

C. Does not matter

# Identical Maximum Lateness

**Claim**: If in an EDF schedule, we swap two jobs with the same deadline, we get the same maximum lateness.

Proof: Since the schedules are EDF, all jobs with the same deadline are scheduled in a consecutive block.
Among those, the last one has the maximum lateness.
That finishing time does not change by swapping schedules within the block.

**Corollary** All EDF schedules have the same maximum lateness.

# Exchange Argument (False Start)

Assume jobs ordered by deadline $d_1 \leq d_2 \leq ... \leq d_n$, so the greedy ordering is simply $A = 1, 2, \ldots, n$. **Claim**: $A$ is optimal

**Proof attempt**: Suppose for contradiction that $A$ is not optimal. Then, there is an optimal solution $O \neq A$.

1. Since $O \neq A$, there must be two jobs $i$ and $j$ that are out of order in $O$.
2. Suppose we could show that swapping the jobs $i$ and $j$ that are out of order gives a *better* solution $O'$.
3. This would mean $O$ is *not* optimal, a contradiction. Therefore, $A$ must be optimal.

**Problem?** We can't show 2. It's true that $O'$ is *no worse* than $O$, but this means $O'$ may still be optimal. Example?

# Exchange Argument (Correct)

Suppose $O$ optimal and $O \neq A$. Then we can modify $O$ to get a new solution $O'$ that is:

1. No worse than $O$
2. Closer to $A$ is some measurable way

$$O(\text{optimal}) \to O'(\text{optimal}) \to O''(\text{optimal}) \to \ldots \to A(\text{optimal})$$

High-level idea: gradually transform an arbitrary optimal solution $O$ into $A$ without hurting solution, thus preserving optimality. Conclude that $A$ is optimal.

Concretely: show 1 and 2 above.

# Exchange Argument for Scheduling to Minimize Lateness

Recall $A = 1, 2, \ldots, n$. For $O \neq A$, say there is an inversion if $i$ comes before $j$ but $j < i$ (thus $d_j \leq d_i$)

**Claim**: if $O$ has an inversion, $O$ has a consecutive inversion—one where $i$ comes immediately before $j$. Why?

**Main result**: let $O \neq A$ be an optimal schedule. Then $O$ has a consecutive inversion $i, j$. We can swap $i$ and $j$ to get a new schedule $O'$ such that:

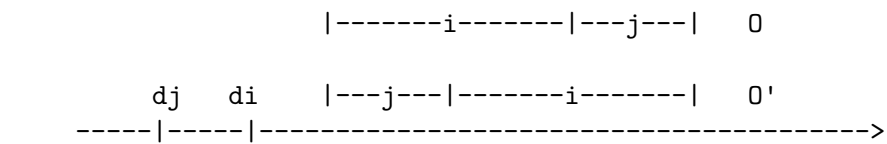
1. $O'$ has one less inversion than $O$
2. Maximum lateness of $O'$ is at most maximum lateness of $O$

**Proof**:

1. Obvious
2. Next slide(s)

# Proof (Lateness does not increase)

Swapping a consecutive inversion ($i$ precedes $j$; $d_j \leq d_i$)




Consider the lateness $\ell'_k$ of each job $k$ in $O'$:

- If $k \notin \{i, j\}$, then lateness is unchanged: $\ell'_k = \ell_k$
- Job $j$ finishes earlier in $O'$ than $O$: $\ell'_j \leq \ell_j$
- Finish time of $i$ in $O'$ = finish time of $j$ in $O$. Therefore

$$\ell'_i = f'_i - d_i = f_j - d_i \leq f_j - d_j = \ell_j$$

**Conclusion**: $\max_k \ell'_k \leq \max_k \ell_k$. Therefore $O'$ is still optimal.

# Wrap-Up (Exchange Argument)

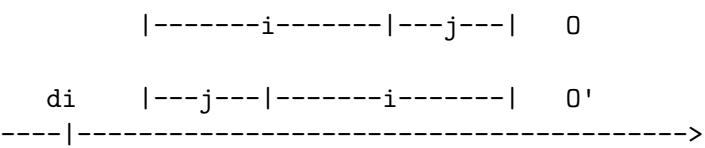
For any optimal $O \neq A$ we showed that we could transform $O$ to $O'$ such that:

1. $O'$ is still optimal
2. $O'$ has one less inversion than $O$

$$O(\text{optimal}) \to O'(\text{optimal}) \to O''(\text{optimal}) \to \ldots \to A(\text{optimal})$$

Since there are at most $\binom{n}{2}$ inversions, by repeating the process a finite number of times we see that $A$ is optimal.

# Clicker




Consider the *total* lateness $\ell'_i + \ell'_j$ in the new schedule. Which fact about total lateness follows from our argument?

A. It is no more than $2\ell_i$
B. It is no more than $\ell_i + \ell_j$
C. It is no more than $2\ell_j$
D. None of the above

B would imply EDF is also optimal for minimizing total lateness. It is not. There is no known polynomial time algorithm for minimizing total lateness.

# Wrap-Up: Greedy Algorithms

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Proof techniques

- Last time: "greedy stays ahead" (inductive proof) ✓
- This time: exchange argument ✓