THE SAMPLE COMPLEXITY OF TOEPLITZ COVARIANCE ESTIMATION

Cameron Musco (Microsoft Research $\rightarrow$ UMass Amherst)

Joint with Yonina Eldar, Jerry Li, and Christopher Musco.

Covariance Estimation Problem. Consider positive semidefinite matrix $T \in \mathbb{R}^{d\times d}$ and distribution $\mathcal{D}$ over d-dimensional vectors with covariance $\mathbb{E}_{x\sim\mathcal{D}}[xx^T] = T$ (i.e., $T_{j,k}$ is the covariance between x_j and x_k).

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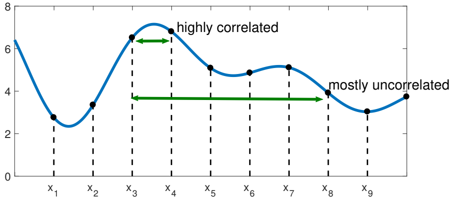
Arises often in signal processing, when measurements are taken on a spatial or temporal grid and covariance depends only on the distance between them – i.e., $\mathbb{E}[x_j \cdot x_k] = f(|j-k|)$.



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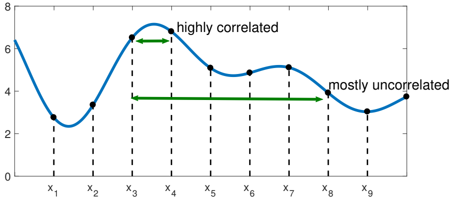

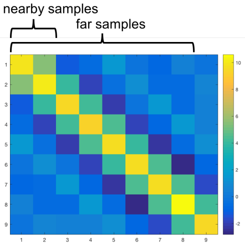

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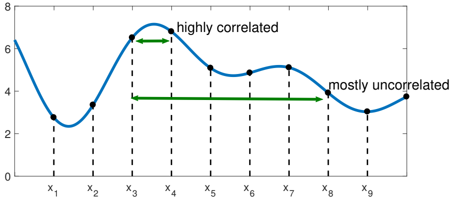

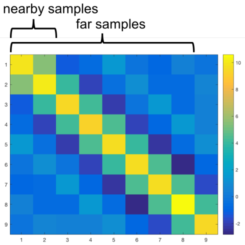

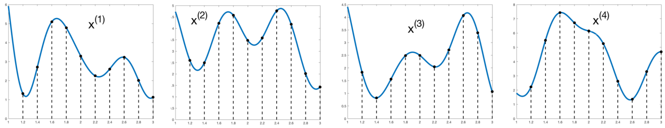

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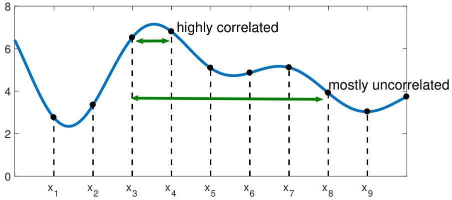

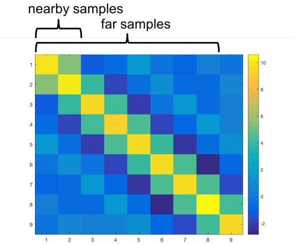

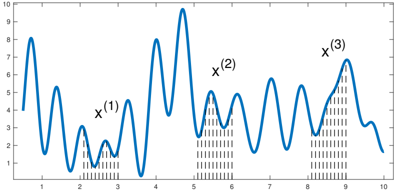

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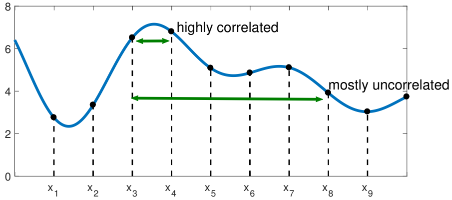

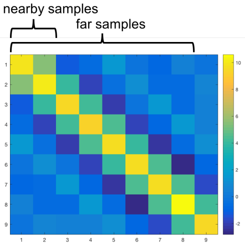

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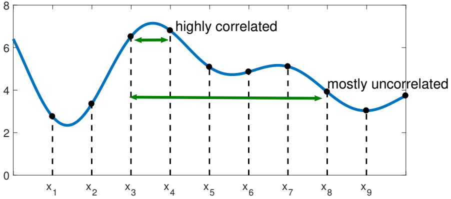

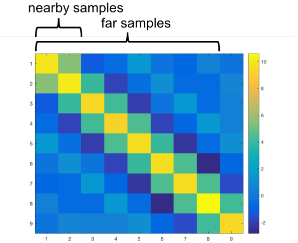

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- Kernel matrices in machine learning are Toeplitz covariance matrices when data points are on a grid.

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- Seems to be interesting even beyond Toeplitz covariance matrices, but not well studied.

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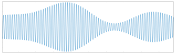
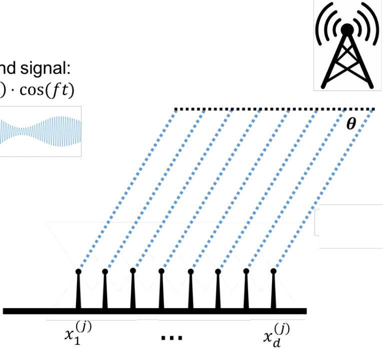
narrowband signal:
$s(t) = a(t) \cdot \cos(ft)$





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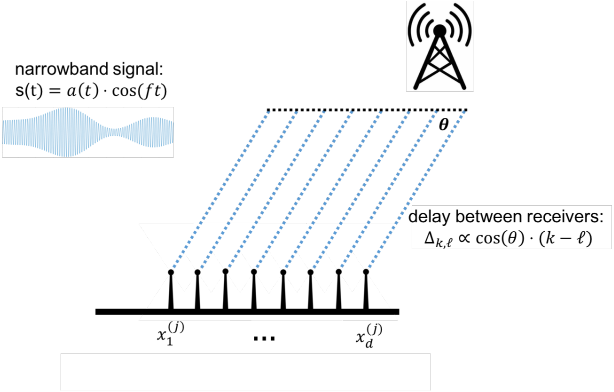

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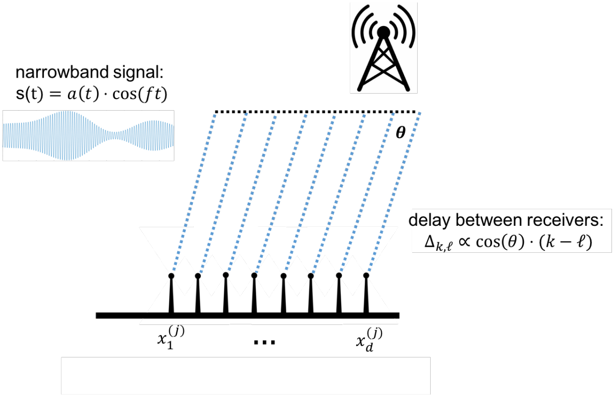

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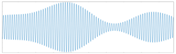

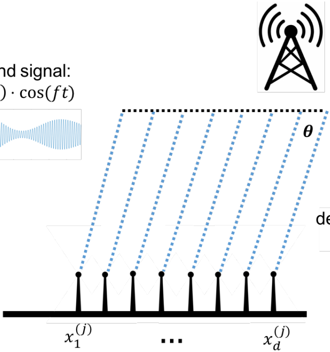

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$\Delta_{k,\ell} \propto \cos(\theta) \cdot (k - \ell)$

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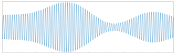

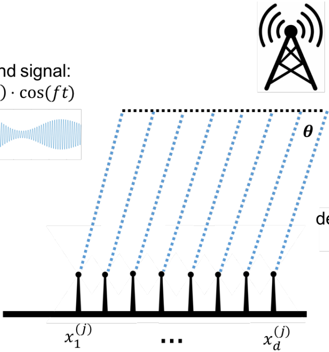

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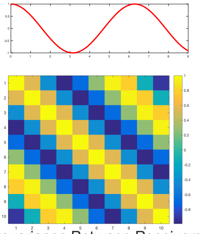

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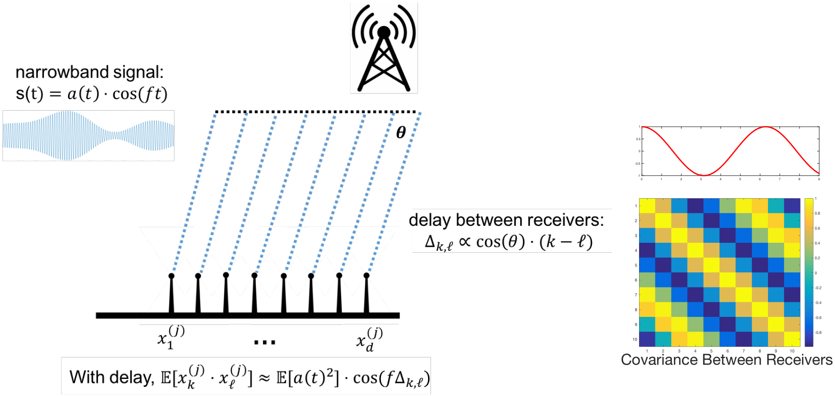

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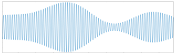

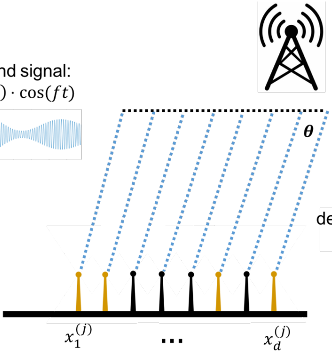

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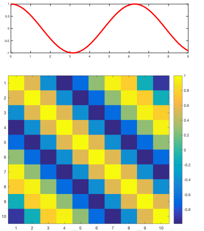
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- Show that sparse ruler methods give sublinear total sample complexity when T is low-rank (e.g., DOA with $k \ll d$ senders).
- Develop improved algorithms in the low-rank setting using techniques from matrix sketching, leverage score-based sampling, and sparse Fourier transforms. Resemble popular 'subspace methods' such as MUSIC and ESPRIT.

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Apply tools from TCS to tackle fundamental signal processing problems. *A Universal Sampling Method for Reconstructing Signals with Simple Fourier Transforms* [AKMMVZ STOC '19]

For today, consider algorithms that sample $x^{(1)}, \ldots, x^{(n)} \sim \mathcal{D}$ with covariance T, read a fixed subset of entries $R \subseteq [d]$ from each $x^{(j)}$, and approximate T using $x_R^{(1)}, \ldots, x_R^{(n)} \in \mathbb{R}^{|R|}$.

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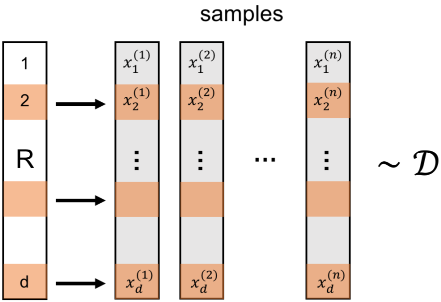

Entry sample complexity: $|R|$. Total sample complexity: $|R| \cdot n$.

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$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{vs.} \quad T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

To notice correlation between x_j and x_k must read both.

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Will see that we can achieve $|R| = O(\sqrt{d})$.

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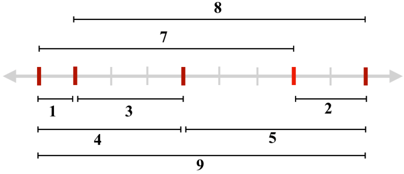
E.g., for $d = 10$, $R = \{1, 2, 5, 8, 10\}$ is a ruler.



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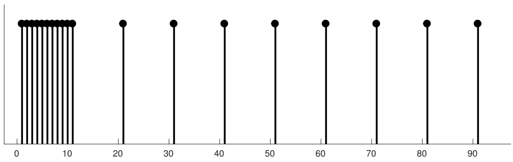

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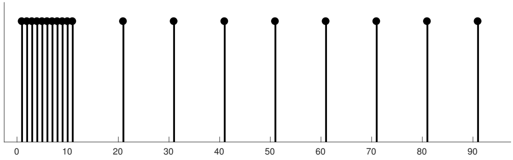

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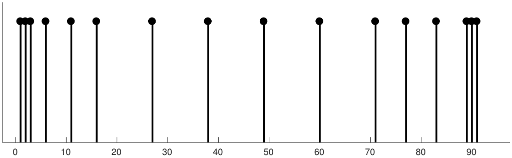

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- If R is a ruler, for each $s \in \{0, \ldots, d-1\}$, there is at least one $k, \ell \in R$ with $|k - \ell| = s$ and thus with covariance
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- With enough samples n from $\mathcal{D}$, will converge on an estimate of each a_s and so of the full matrix T.

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- How does the total sample complexity compare to methods that read every entry of each $x^{(j)}$, e.g., estimating T with the empricial covariance $\hat{T} = \frac{1}{n}\sum_j x^{(j)}x^{(j)^T}$.

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Theorem. For any ruler $R \subset [d]$, covariance estimation with R gives $\|\tilde{T} - T\|_2 \leq \varepsilon \|T\|_2$ with entry sample complexity $|R|$ and vector sample complexity $n = \tilde{O}\left(\frac{d}{\varepsilon^2}\right)$.

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- Proof uses the Fourier structure of Toeplitz matrices.

Algorithm: For each $s \in \{0, 1\}$ approximate a_s by average over the ruler R:

$$\tilde{a}_s = \frac{1}{n|R_s|} \sum_{j=1}^{n} \sum_{(k,\ell) \in R_s} x_k^{(j)} \cdot x_\ell^{(j)} \text{ where } R_s = \{k, \ell \in R : |k - \ell| = s\}.$$

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- Let $E = T - \tilde{T}$ and $e = a - \tilde{a}$. We want to bound $\|E\|_2$.

Entry approximation to matrix approximation: Can bound $\|\tilde{T} - T\|_2 = \|E\|_2$ in terms of the Fourier transform of e.

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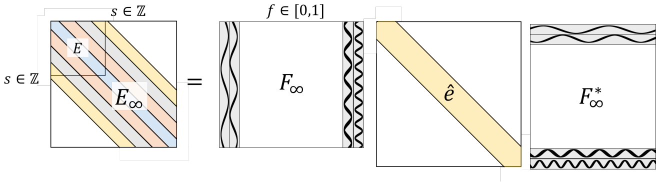

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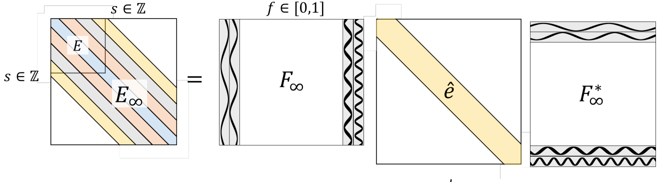

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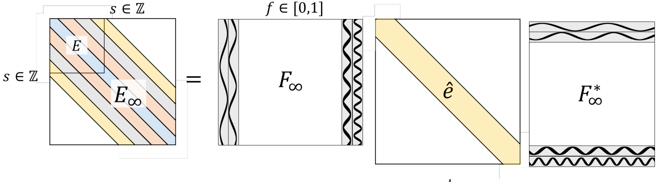

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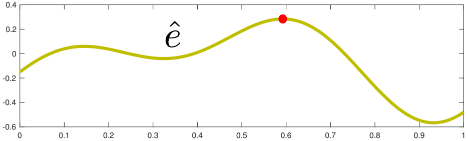

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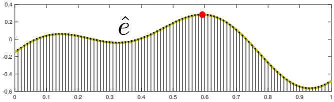

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- Setting $\varepsilon' = \varepsilon/\sqrt{d}$ and union bounding over a net of f values gives our $n = \tilde{O}(d/\varepsilon^2)$ bound.
- The more *coverage* R has (the larger the $|R_s|$ is on average), the smaller $\|M_f\|_F$ will be. Let's us interpolate between minimal entry sample complexity and minimal vector sample complexity.

FULL RULER SAMPLE COMPLEXITY

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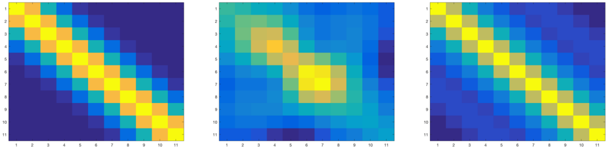

True covariance T | Empirical covariance $\hat{T}$ | Improved estimator $avg(\hat{T})$

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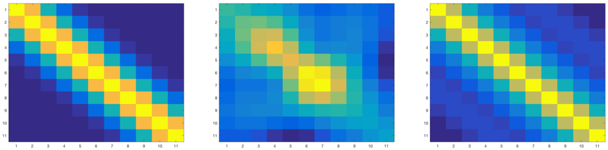

True covariance T Empirical covariance $\hat{T}$ Improved estimator $avg(\hat{T})$

- Improves on sample complexity of just using the empirical covariance by a $\tilde{O}(d)$ factor.

Total sample complexity is $O(\sqrt{d}) \cdot \tilde{O}(d) = \tilde{O}(d^{3/2})$ for sparse ruler vs. $d \cdot \tilde{O}(1) = \tilde{O}(d)$ for full sample estimation.

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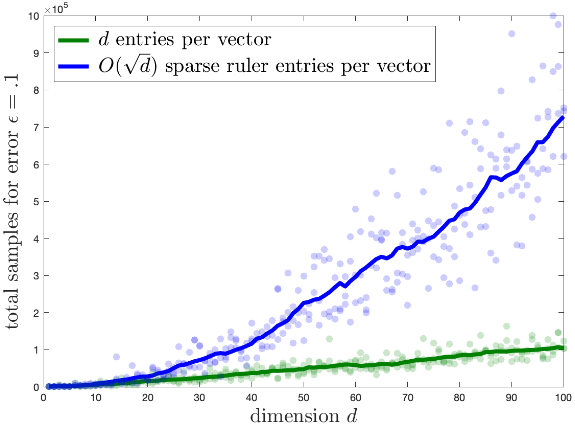

SPARSE RULER VS. FULL RULER

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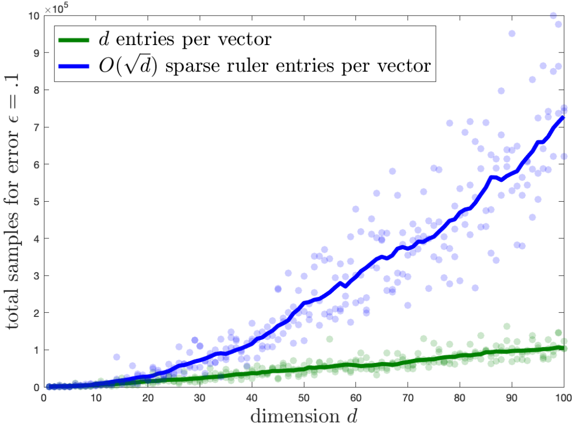
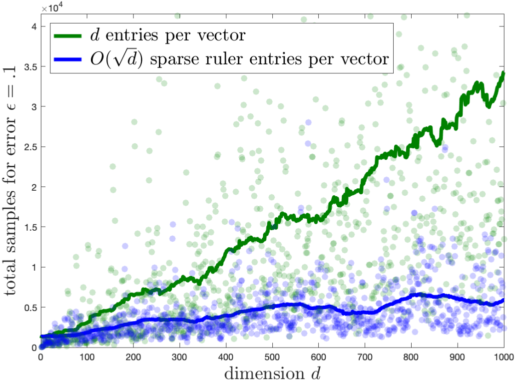
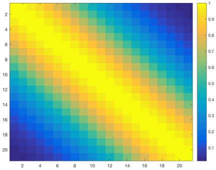
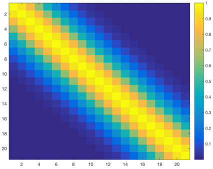


- Prove bounds are tight when T is the identity.

IS THERE ALWAYS A TRADEOFF?







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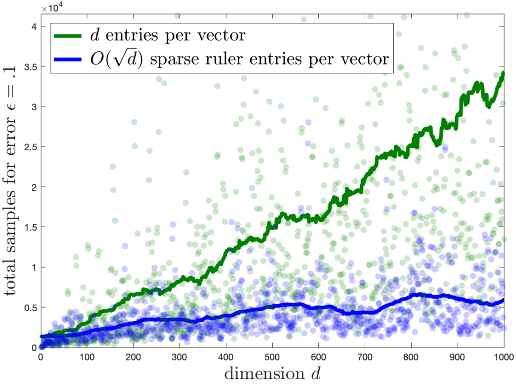
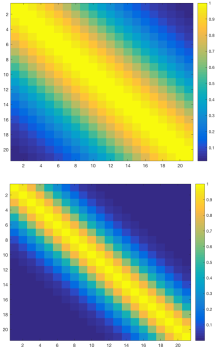
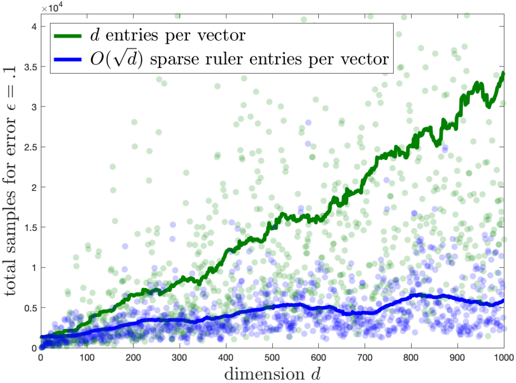
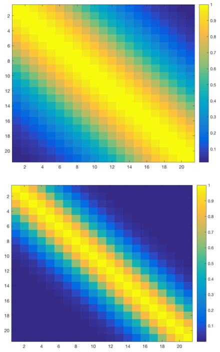




- Total sample complexity is $\tilde{O}(\sqrt{d})$ for sparse ruler estimation vs. $\tilde{O}(d)$ for full sample estimation.





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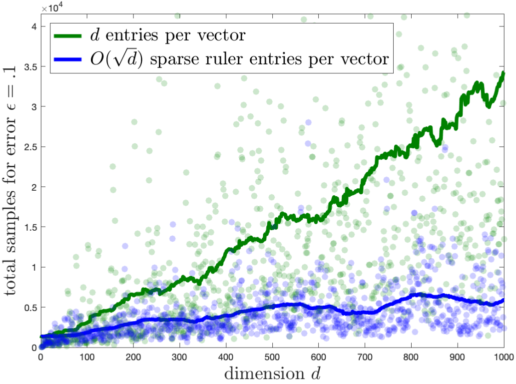

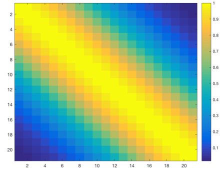

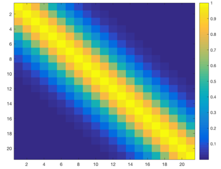

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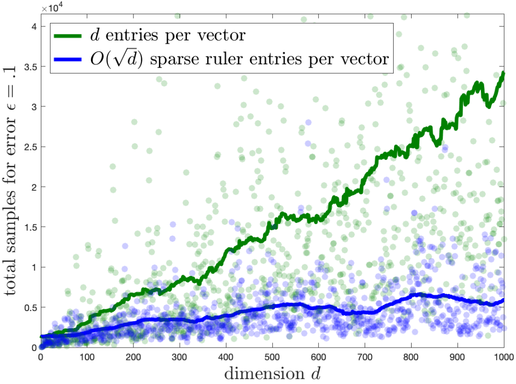

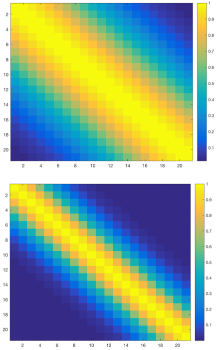

- Total sample complexity is $\tilde{O}(\sqrt{d})$ for sparse ruler estimation vs. $\tilde{O}(d)$ for full sample estimation.
- Sparse rulers give much better total sample complexity when T is (approximately) low-rank. **Can we explain this?**

Recall that we have with $n = \tilde{O}(1/\varepsilon^2)$ samples:

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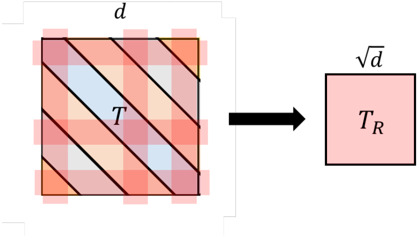

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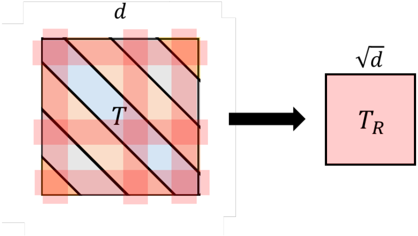

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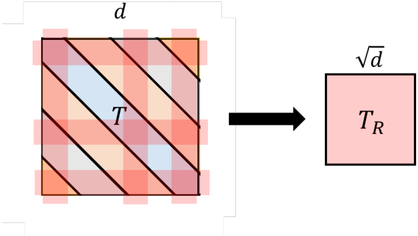

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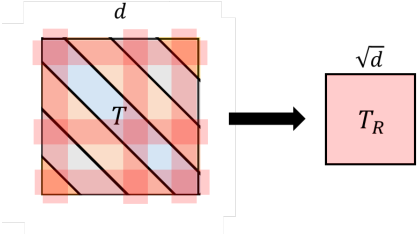

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- Low-rank matrices cannot look like the identity – have significant off diagonal mass [MMW '19].
- **Upshot**: Show $\|T_R\|_2 \leq \frac{k}{\sqrt{d}}\|T\|_2$. Setting $\varepsilon' = \varepsilon/k$ obtain total sample complexity $\tilde{O}\left(\frac{\sqrt{d}k^2}{\varepsilon^2}\right)$.

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- Connections between these two approaches.

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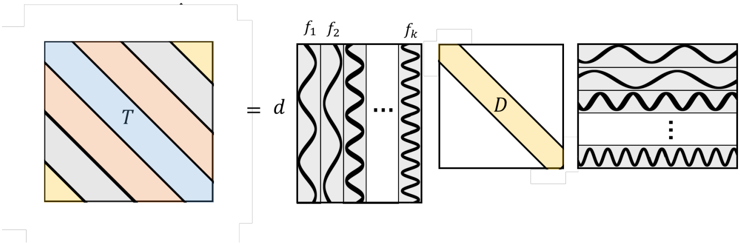

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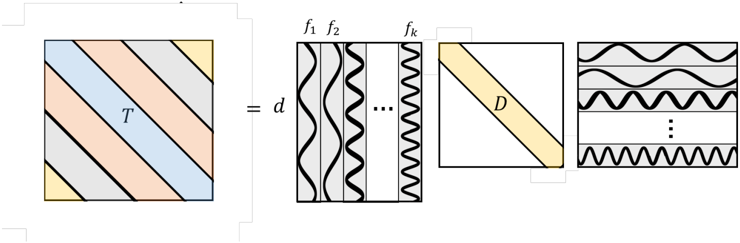

- Any sample $x \sim \mathcal{N}(0, T)$ can be written as $F_S D^{1/2} g$ for $g \sim \mathcal{N}(0, I)$. $\mathbb{E}[xx^T] = F_S D^{1/2} \mathbb{E}[gg^T] D^{1/2} F_S^* = T$.

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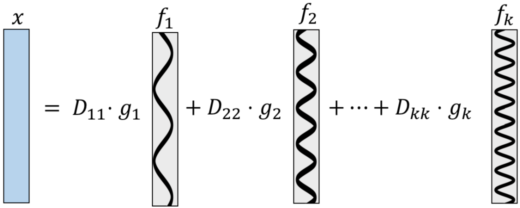

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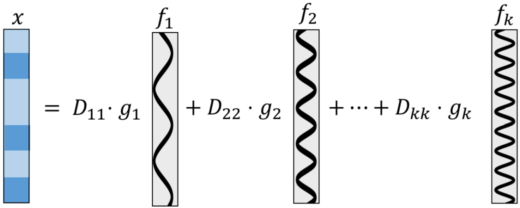

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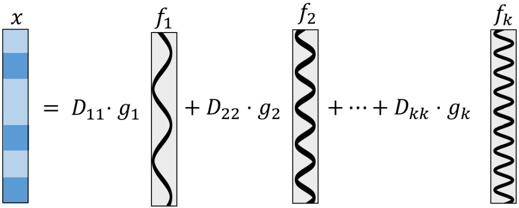

- Can recover exactly e.g. via Prony's sparse Fourier transform method by reading any $2k$ entries.
- Take $n = \tilde{O}(1/\varepsilon^2)$ samples, recover each in full by reading $2k$ entries, and then apply our earlier resut for full ruler $R = [d]$. Total sample complexity: $\tilde{O}(k/\varepsilon^2)$.

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- Well studied in TCS, especially in the case when $f_1, \ldots, f_k$ are 'on grid' integer frequencies.

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  Theorem: Any $A \in \mathbb{R}^{n \times d}$, contains a subset of $O(k/\varepsilon)$ columns, C such that:

$$\|A - P_C \cdot A\|_F^2 \leq (1+\varepsilon) \min_{\text{rank}-k\ M} \|A - M\|_F^2.$$

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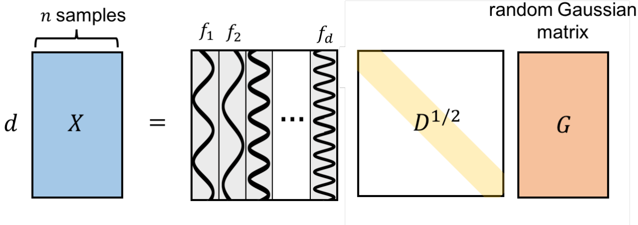

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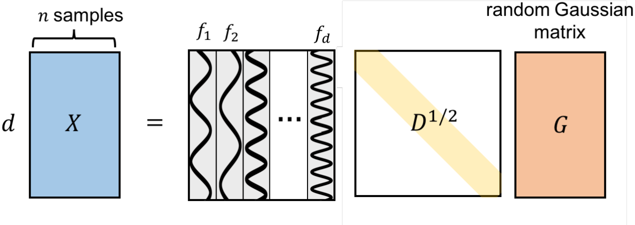

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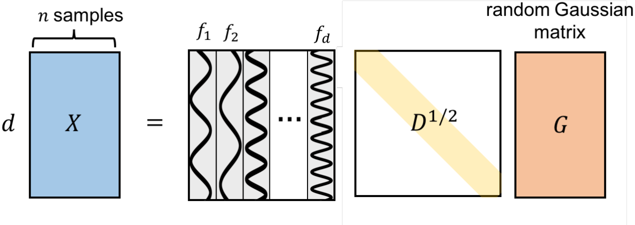

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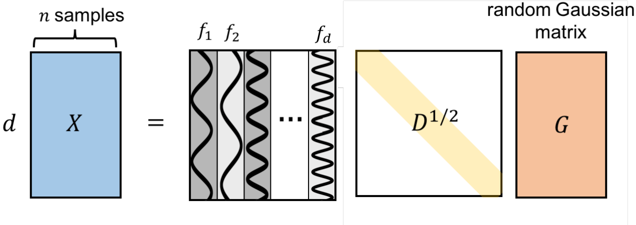

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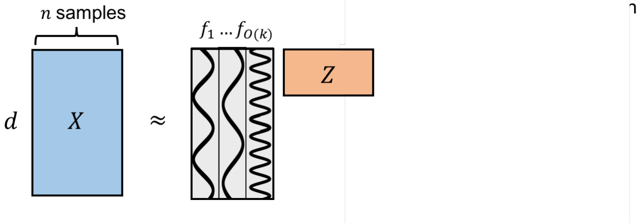

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- Find frequencies via brute force search over a net.
- At each step of the search, for a given F_M, we must find Z that reconstructs X as well as possible using these frequencies. How do we do this without reading all of X?

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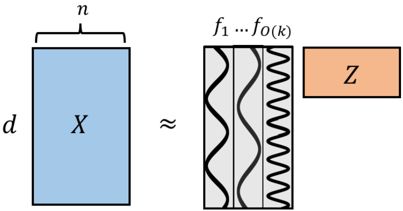

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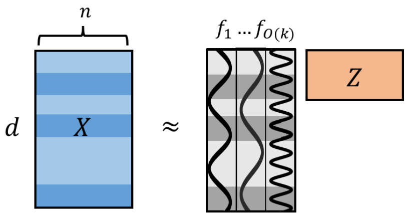

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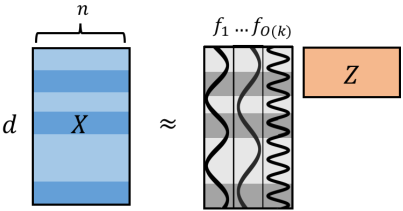

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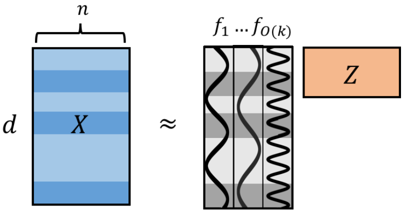

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- **Remark:** If $f_1, \ldots, f_m$ are 'on-grid' integers, the columns of F_M are orthonormal and the leverage scores are all $k/n \rightarrow$ RIP for subsampled Fourier matrices.

Leverage scores measure much large a function in the column span of F_M can be at index i (i.e., how important that index may be in the regression.)

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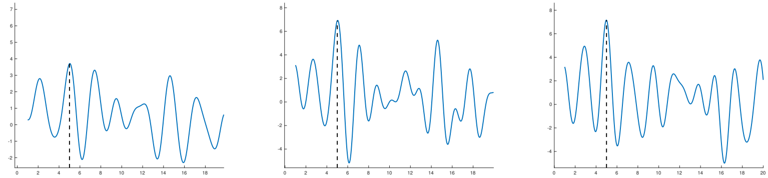

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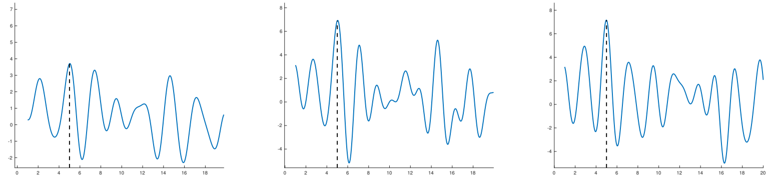
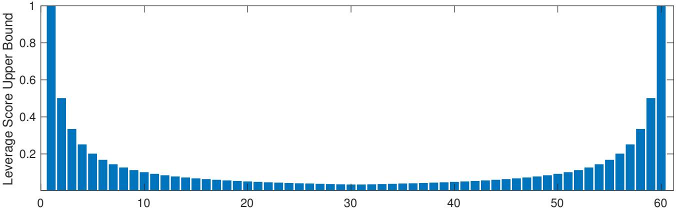


- Using that $F_M y$ is a Fourier sparse function we can bound this quantity a priori, without any dependence on F_M.

Extend bounds of [Chen Kane Price Song '16] to give explicit function upper bounding the leverage scores of any F_M:



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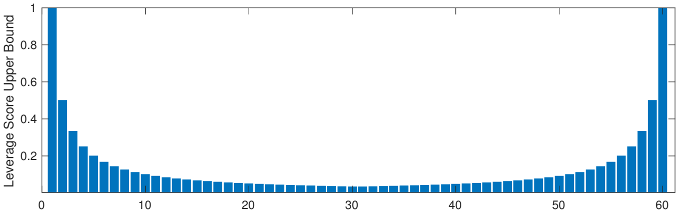

Since this distribution is universal, can sample one set of entries by these leverages scores, and find $X \approx F_M \cdot Z$ with high probability for any set of frequencies $f_1, \ldots, f_m$ in net.

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Sample Complexity: Gives $\|T - \tilde{T}\|_2 \leq \varepsilon \|T\|_2 + f(T - T_k)$ when X contains $n = \tilde{O}(\text{poly}(k/\epsilon))$ samples. Entry sample complexity poly(k/ε), total sample complexity $\tilde{O}(\text{poly}(k/\varepsilon))$.

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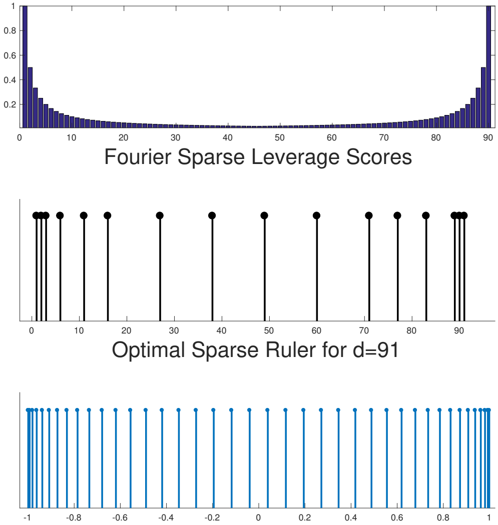
  - Sample complexity bounds and tradeoffs for applications like direction-of-arrival estimation, Doppler imaging.

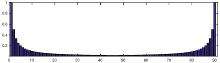
CONNECTIONS BETWEEN SAMPLING SCHEMES



Fourier Sparse Leverage Scores

Optimal Sparse Ruler for d=91

Degree 40 Chebyshev Nodes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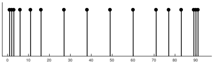
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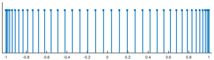
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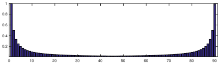

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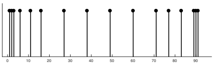

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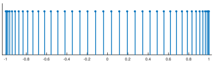
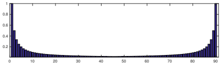


Degree 40 Chebyshev Nodes

- Some Formal Connections:
  - Limiting density of Chebyshev nodes is the leverage score distribution for k degree polynomials.



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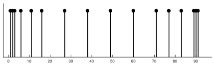
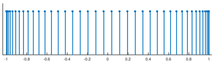
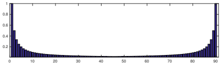


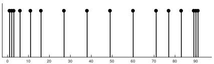
Optimal Sparse Ruler for d=91

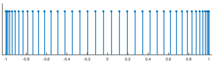


Degree 40 Chebyshev Nodes

- **Some Formal Connections:**
  - Limiting density of Chebyshev nodes is the leverage score distribution for k degree polynomials.
  - Sampling $O(\sqrt{d})$ indices via Fourier sparse leverage scores gives a sparse ruler with good probability.


Fourier Sparse Leverage Scores


Optimal Sparse Ruler for d=91


Degree 40 Chebyshev Nodes

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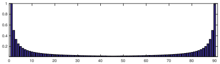
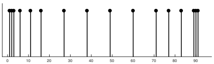
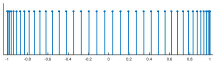

Fourier Sparse Leverage Scores


Optimal Sparse Ruler for d=91


Degree 40 Chebyshev Nodes

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- Seem to have a lot more to understand.

Thanks! Questions?

Paper draft and slides available at `cameronmusco.com`