SPECTRUM APPROXIMATION BEYOND FAST MATRIX MULTIPLICATION: ALGORITHMS AND HARDNESS

Cameron Musco (MIT), Praneeth Netrapalli (MSR), Aaron Sidford (Stanford), Shashanka Ubaru (UMN), David Woodruff (CMU)

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MOTIVATING QUESTION

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- Most can be solved with 'heavy hammers' like matrix inversion, full eigendecomposition, and SVD. These computations are known to be equivalent in difficulty to matrix multiplication and take $O(n^\omega)$ time for $n \times n$ matrices, $O(n^3)$ time in practice.

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[1] Important even if $\omega = 2$.

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- **Coarser approximation** methods give general solutions e.g. for linear systems, eigenvalue computation, and low-rank approximation, with $\text{poly}(1/\epsilon)$ dependence.
- **Essentially nothing is known beyond these techniques.**
- All known linear algebraic algorithms which work with high accuracy on general matrices require full $n \times n \times n$ matrix multiplication (i.e. $O(n^\omega)$ time). Why?

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- No reduction known for uniform computation.
- An emerging line of work on reductions and hardness for linear algebraic problems [Kyng, Zhang '17], [Backurs, Indyk, Schmidt '17], [Musco, Woodruff '17].

THIS WORK

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- Bounds extend to many natural problems like determinant, trace inverse, effective resistance computation, etc.

Basic Question: Given a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, can we approximate, in some useful way, its singular value spectrum $\sigma_1 > .... > \sigma_n \geq 0$ without performing a full SVD. I.e., in $o(n^\omega)$ time, for the current value of ω?

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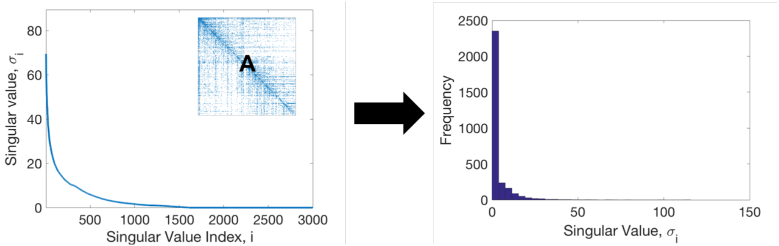
Our algorithmic contribution: Show how to efficiently compute an approximate histogram of the spectrum.



APPLICATION: $o(n^\omega)$ TIME MATRIX NORMS

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- $\tilde{O}(n^2 \cdot p/\epsilon^3)$ time algorithm for approximating the Schatten p-norm for any real $p > 2$:

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$$\|\mathbf{A}\|_p = \left(\sum_{i=1}^{n} \sigma_i^p\right)^{1/p}.$$
- Results for general Schatten p-norms, SVD entropy, and more general matrix norms of the form $\sum_{i=1}^{n} g(\sigma_i)$.

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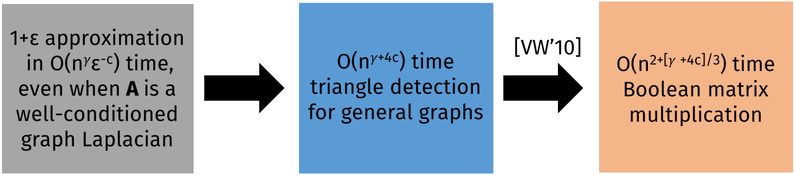

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1+ε approximation in $O(n^{\gamma}\varepsilon^{-c})$ time, even when A is a well-conditioned graph Laplacian

$\Rightarrow$

$O(n^{\gamma+4c})$ time triangle detection for general graphs

$\Rightarrow$ [VW'10]

$O(n^{2+[\gamma+4c]/3})$ time Boolean matrix multiplication

- Our $\tilde{O}(n^2/\epsilon^3)$ time algorithm for $\|\mathbf{A}\|_3$ would give faster triangle detection if the ϵ dependence was $\approx \frac{1}{\epsilon^{1/10}}$.

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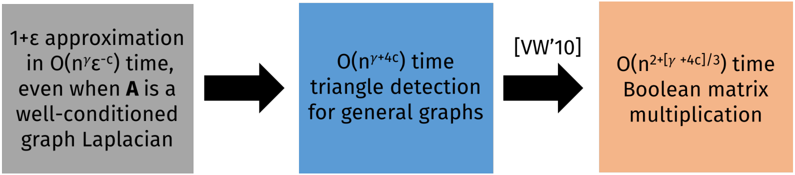

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- An $O(n^{3-\delta} \cdot \log(1/\epsilon))$ time algorithm gives $\tilde{O}(n^{3-\delta})$ time triangle detection and $\tilde{O}(n^{3-\delta/3})$ time matrix multiplication.

HIGH LEVEL VIEW

Slow: $\Theta(n^\omega)$

Matrix multiplication
Matrix inversion
Eigendecomposition
Full SVD

Fast, Approximate:
$o(n^\omega)/\varepsilon^c$ for $c \geq 1/2$

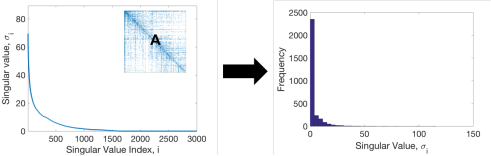
Linear systems
Top eigenvalue
Low-rank approximation
Schatten norms
SVD entropy

Fast, With Assumptions:

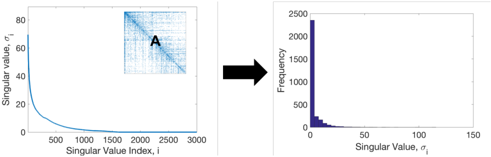
Linear systems
Eigenvectors/values
Low-rank approximation
$\exp(\mathbf{A})$, $\mathbf{A}^{1/2}$, etc.

Fast, Accurate, No Assumptions:
$o(n^\omega)/\varepsilon^c$ for small c
$o(n^\omega \log(1/\varepsilon))$

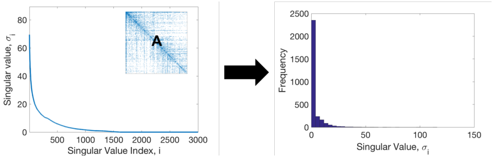
Anything?
Our results give negative evidence for many candidate problems.



Singular value, σ_i
Singular Value Index, i
A
Frequency
Singular Value, σ_i



Key Primitive: Approximately count the number of singular values in each bucket.



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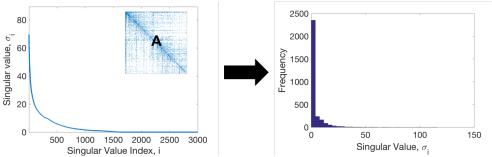
- Combine randomized trace estimation, stochastic optimization, polynomial approximation, and preconditioning.



Key Primitive: Approximately count the number of singular values in each bucket.

- Combine randomized trace estimation, stochastic optimization, polynomial approximation, and preconditioning.
- Leverage stochastic gradient based system solvers, which give better guarantees than the conjugate gradient method for certain spectrums. Use these guarantees to give generic speed ups. E.g., $O(n^{2.5}) \to O(n^{2.33})$ for $\|\mathbf{A}\|_*$.

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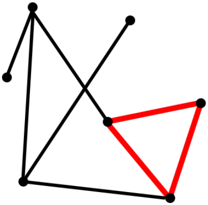

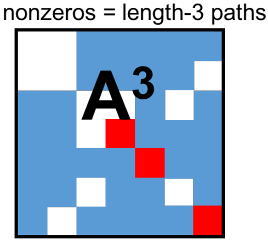

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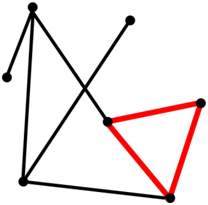
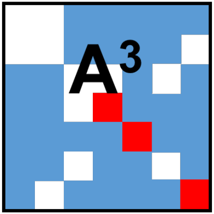


nonzeros = length-3 paths



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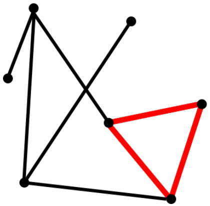

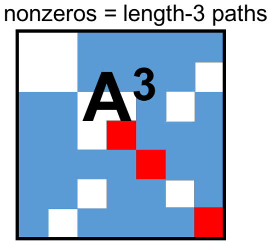

- $\mathrm{tr}(\mathbf{A}^3) = \sum_{i=1}^{n} \lambda_i(\mathbf{A}^3) = \sum_{i=1}^{n} \lambda_i(\mathbf{A})^3 \neq \sum_{i=1}^{n} \sigma_i(\mathbf{A})^3 \overset{\text{def}}{=} \|\mathbf{A}\|_3^3$.

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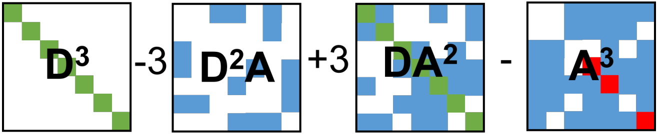

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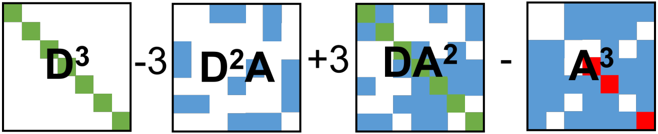

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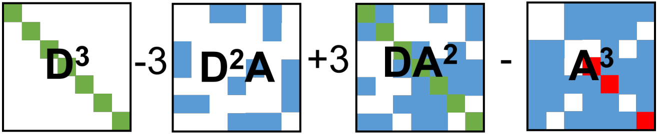

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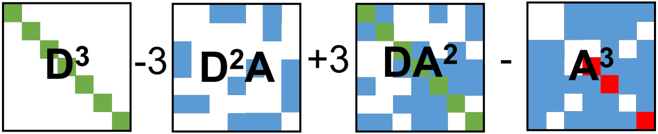

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REDUCTION SKETCH

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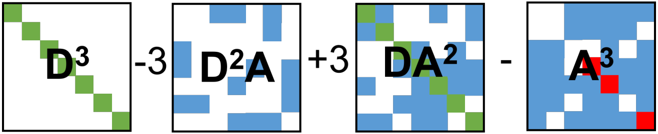

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- So multiplicative $(1+\epsilon)$ approximation for $\epsilon < \frac{1}{8n^4}$ gives triangle detection.
- Computing $(1 \pm \epsilon)\|\mathbf{L}\|_3^3$ in $O(n^\gamma \cdot \epsilon^{-c})$ time gives triangle detection in $O(n^{\gamma+4c})$ time.

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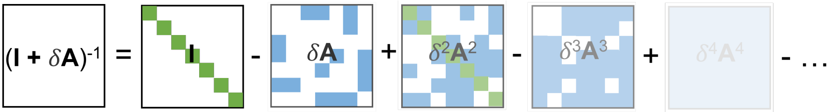

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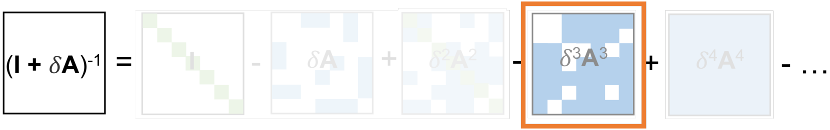
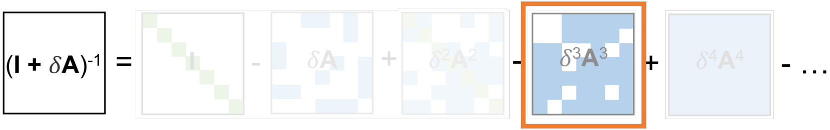


- Traces of large terms can again be computed exactly.
- After subtracting out these terms, $\mathrm{tr}(\delta^3 \mathbf{A}^3)$ dominates $\mathrm{tr}((\mathbf{I} + \delta \mathbf{A})^{-1})$ (we set $\delta = 1/\operatorname{poly}(n)$). So fine enough approximation gives triangle detection.

EXTENDING TO OTHER SPECTRAL SUMMARIES

Extend reduction to other matrix norms, logdet, trace inverse, trace exponential, etc. via Taylor expansion.

- $\frac{1}{1+x} = \sum_{i=0}^{\infty}(-x)^i$.



- Traces of large terms can again be computed exactly.
- After subtracting out these terms, $\mathrm{tr}(\delta^3\mathbf{A}^3)$ dominates $\mathrm{tr}((\mathbf{I}+\delta\mathbf{A})^{-1})$ (we set $\delta = 1/\operatorname{poly}(n)$). So fine enough approximation gives triangle detection.
- Bound for determinant is similar, by expanding out $\det(\mathbf{I}+\delta\mathbf{A}) = \prod_{i=1}^{n}(1+\delta\lambda_i(\mathbf{A}))$.

OPEN QUESTIONS (A SMALL SUBSET)

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Thanks! Questions?