SUBLINEAR TIME LOW-RANK APPROXIMATION OF POSITIVE SEMIDEFINITE MATRICES

Cameron Musco (MIT) and David P. Woodruff (CMU)

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- **Concrete:** Significantly improves on previous, roughly linear time approaches for general matrices, and bypasses a trivial linear time lower bound for general matrices.
- **High Level:** Demonstrates that PSD structure can be exploited in a much stronger way than previously known for low-rank approximation. Opens the possibility of further advances in algorithms for PSD matrices.

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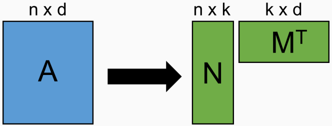

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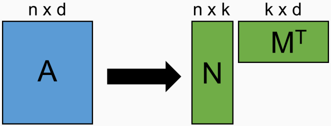

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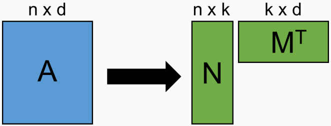

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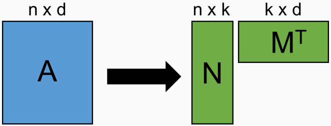

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- Includes graph Laplacians, Gram matrices and kernel matrices, covariance matrices, Hessians for convex functions.

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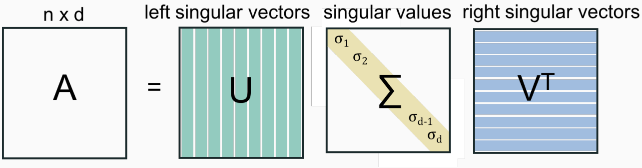

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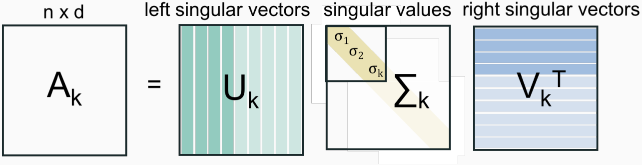

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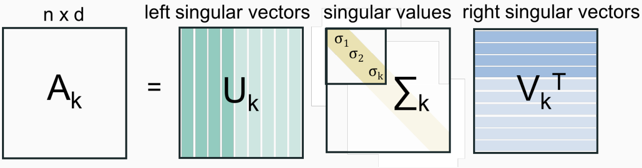

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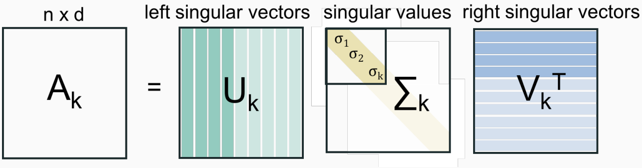

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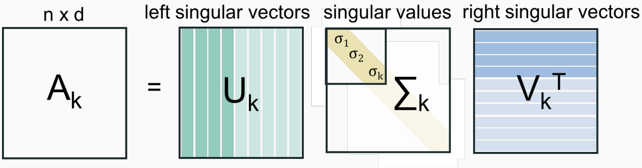

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- Unfortunately, computing the SVD takes $O(nd^2)$ time.

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Theorem (Clarkson, Woodruff '13)

There is an algorithm which in $O(\text{nnz}(\mathbf{A}) + n \cdot \text{poly}(k, 1/\epsilon))$ *time outputs* $\mathbf{N} \in \mathbb{R}^{n \times k}, \mathbf{M} \in R^{d \times k}$ *satisfying with prob.* 99/100*:*

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- Best known runtime for both general and PSD matrices.

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There is an algorithm running in $\tilde{O}\left(\frac{nk^2}{\epsilon^4}\right)$ *time which, given PSD* $\mathbf{A}$, *outputs* $\mathbf{N}, \mathbf{M} \in \mathbb{R}^{n \times k}$ *satisfying with probability* $99/100$*:*

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- Compare to CW'13 which takes $O(\text{nnz}(\mathbf{A})) + n \cdot \text{poly}(k, 1/\epsilon)$.
- If $k, 1/\epsilon$ are not too large compared to $\text{nnz}(\mathbf{A})$, our runtime is significantly sublinear in the size of $\mathbf{A}$.

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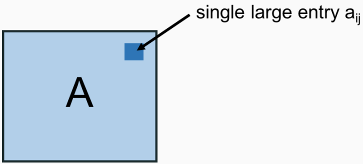

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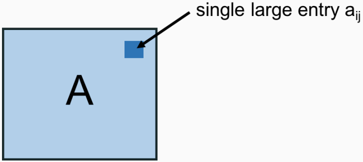

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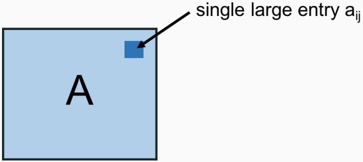

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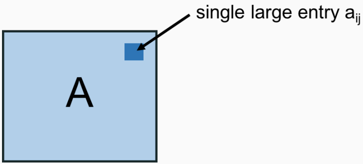

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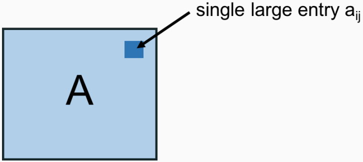

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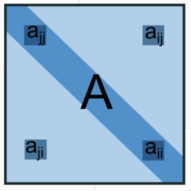
$$\|\mathbf{A} - \mathbf{N}\mathbf{M}^T\|_F \leq \|\mathbf{A} - \mathbf{A}_k\|_F + \epsilon\|\mathbf{A}\|_F.$$

Observation: For PSD $\mathbf{A}$, we have for any entry $\mathbf{a}_{ij}$:

$$\mathbf{a}_{ij} \leq \max(\mathbf{a}_{ii}, \mathbf{a}_{jj})$$

since otherwise $(\mathbf{e}_i - \mathbf{e}_j)^T \mathbf{A}(\mathbf{e}_i - \mathbf{e}_j) < 0$.



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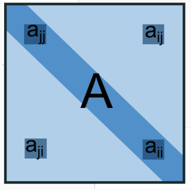
- So we can find any 'hidden' heavy entry by looking at its corresponding diagonal entries.



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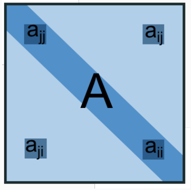

Question: How can we exploit additional structure arising from positive semidefiniteness to achieve sublinear runtime?

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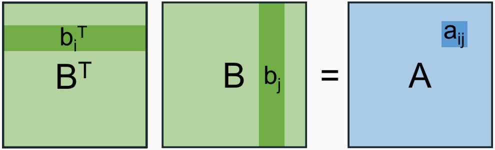
- Letting $\mathbf{b}_1, ..., \mathbf{b}_n$ be the columns of $\mathbf{B}$, the entries of $\mathbf{A}$ contain every pairwise dot product $\mathbf{a}_{ij} = \mathbf{b}_i^T\mathbf{b}_j$.



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- The heavy diagonal observation is just one example. By Cauchy-Schwarz:

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Another View: $\mathbf{A}$ contains a lot of information about the column span of $\mathbf{B}$ in a very compressed form – with every pairwise dot product stored as $\mathbf{a}_{ij}$.

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- Things will be messier once we introduce approximation, but this simple idea will lead to a sublinear time algorithm for $\mathbf{A}$.

Theorem (Deshpande, Vempala '06)

For any $\mathbf{B} \in \mathbb{R}^{n \times n}$, *there exists a subset of* $\tilde{O}(k^2/\epsilon)$ *columns whose span contains* $\mathbf{Z} \in \mathbb{R}^{n \times k}$ *satisfying:*

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Adaptive Sampling

Initially, start with an empty column subset $\mathcal{S} := \{\}$.

For $t = 1, ..., \tilde{O}(k^2/\epsilon)$

Let $\mathbf{P}_{\mathcal{S}}$ be the projection onto the columns in $\mathcal{S}$.

Add $\mathbf{b}_i$ to $\mathcal{S}$ with probability $\frac{\|\mathbf{b}_i - \mathbf{P}_{\mathcal{S}}\mathbf{b}_i\|^2}{\sum_{i=1}^{n}\|\mathbf{b}_i - \mathbf{P}_{\mathcal{S}}\mathbf{b}_i\|^2}$.

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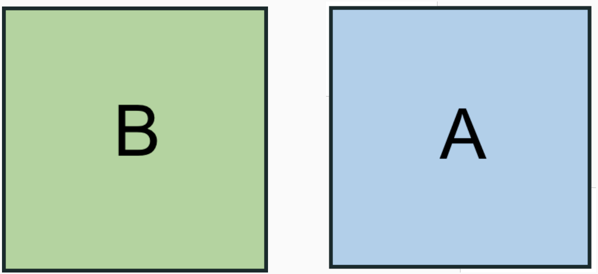

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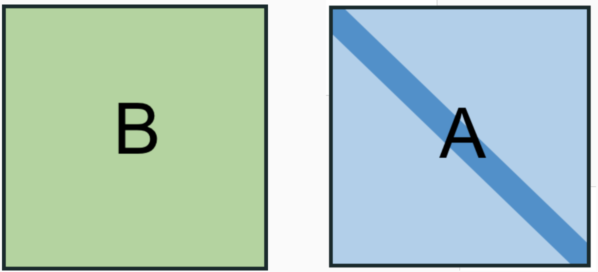

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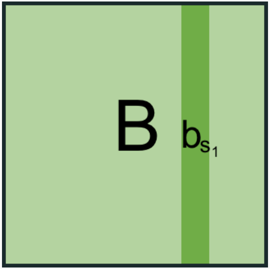

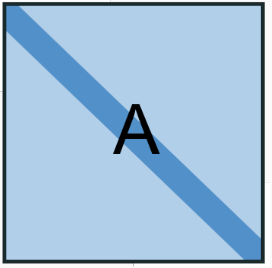

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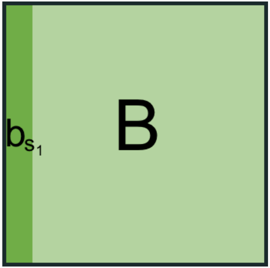

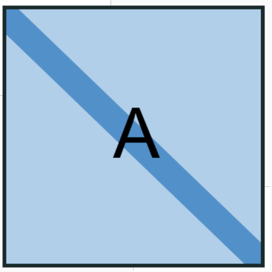

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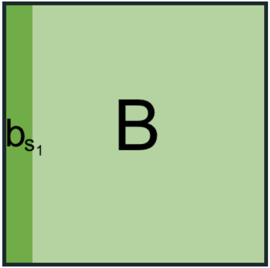

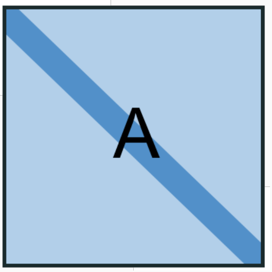

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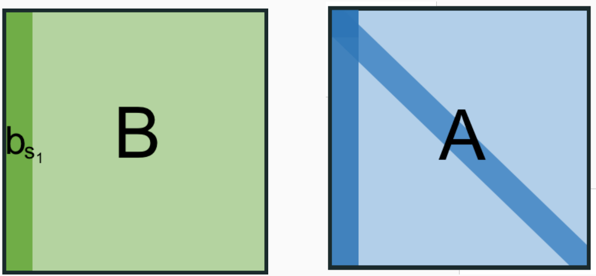

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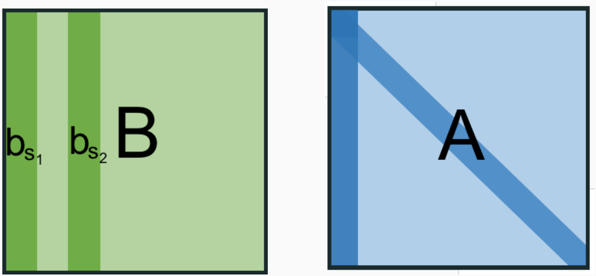

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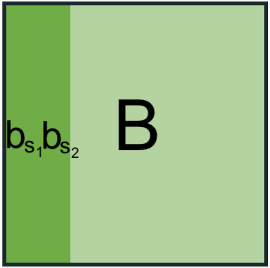

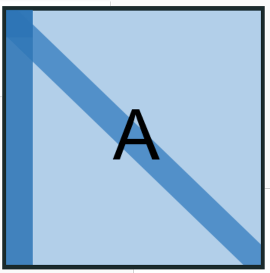

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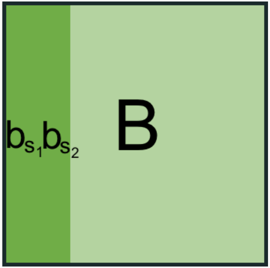

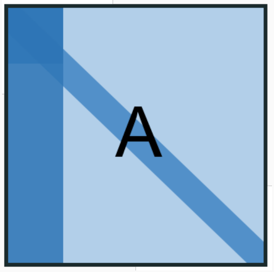

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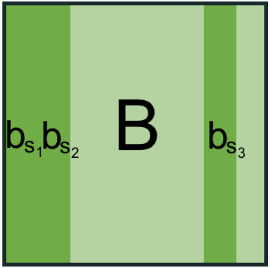

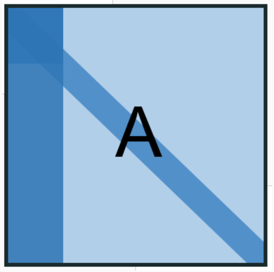

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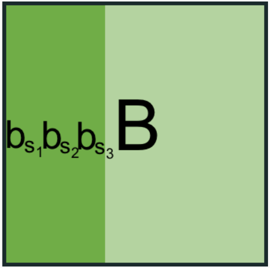

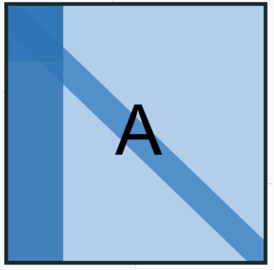

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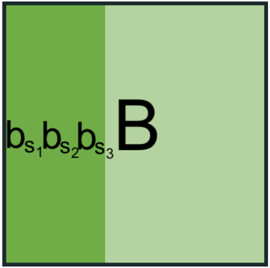

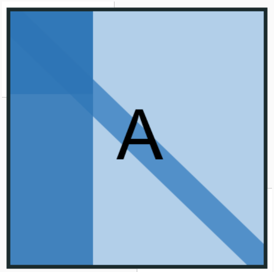

Theorem (Factor Matrix Low-Rank Approximation)

There is an algorithm using $\tilde{O}(nk^2/\epsilon)$ accesses to $\mathbf{A} = \mathbf{B}^T\mathbf{B}$ which computes $\mathbf{Z} \in \mathbb{R}^{n \times k}$ satisfying with probability $99/100$:

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- How does this translate to low-rank approximation of $\mathbf{A}$ itself?

Lemma

If $\|\mathbf{B} - \mathbf{Z}\mathbf{Z}^T\mathbf{B}\|_F^2 \leq \left(1 + \frac{\epsilon^{3/2}}{\sqrt{n}}\right) \|\mathbf{B} - \mathbf{B}_k\|_F^2$ *, then for* $\mathbf{A} = \mathbf{B}^T\mathbf{B}$*:*

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Boosting to a PSD matrix approximation

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- Our best algorithm accesses just $\tilde{O}\left(\frac{nk}{\epsilon^{2.5}}\right)$ entries of $\mathbf{A}$ and runs in $\tilde{O}\left(\frac{nk^2}{\epsilon^4}\right)$ time.

LIMITATIONS OF COLUMN SAMPLING

Recall that our algorithm accesses the diagonal of **A** along with $\tilde{O}(k^2/\epsilon)$ columns.

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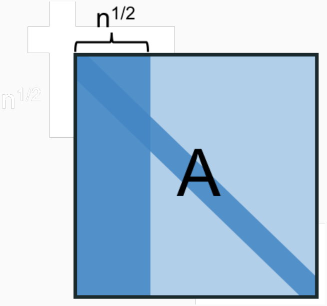

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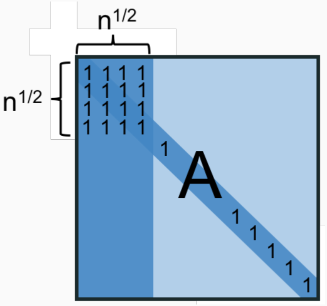

- If we take fewer columns, we can miss a $\sqrt{n} \times \sqrt{n}$ block which contains a constant fraction of $\mathbf{A}$'s Frobenius norm.

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COLUMN AND ROW SAMPLING

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- Same intuition – select a diverse set of columns which span a near-optimal low-rank approximation of the matrix.
- Sample $\mathbf{AS}$ is a projection-cost-preserving sketch for $\mathbf{A}$ [Cohen et al '15,'17]. For any rank-k projection $\mathbf{P}$,

$$\|\mathbf{AS} - \mathbf{PAS}\|_F^2 = (1 \pm \epsilon)\|\mathbf{A} - \mathbf{PA}\|_F^2.$$

FINAL ALGORITHM

Recover low-rank approximation using two-sided sampling and projection-cost-preserving sketch property.

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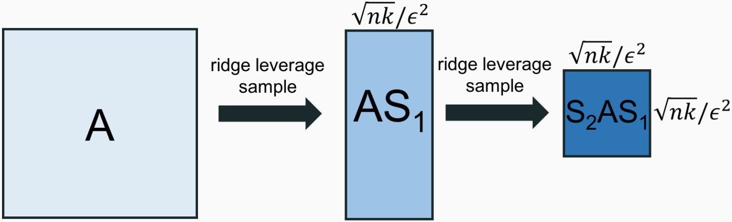

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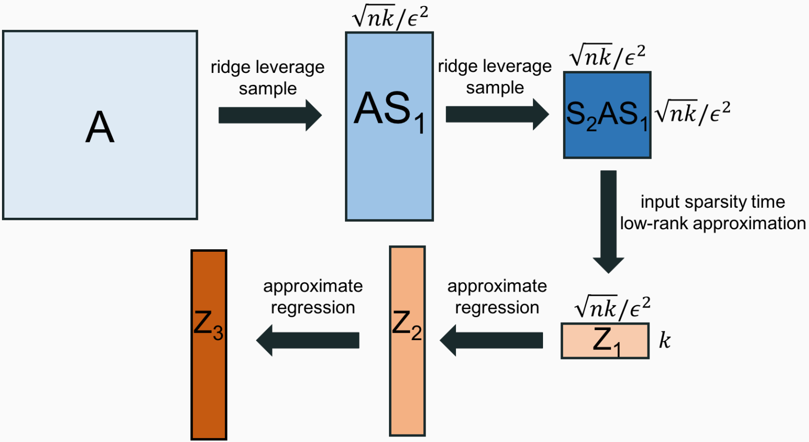

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- Since $\mathbf{B}$ has the same singular vectors as $\mathbf{A}$ and $\sigma_i(\mathbf{B}) = \sqrt{\sigma_i(\mathbf{A})}$, a low-rank approximation of $\mathbf{B}$ can used to find one for $\mathbf{A}$, albiet with a $\sqrt{n}$ factor loss in quality.

SUMMARY OF MAIN IDEAS

- View each entry of $\mathbf{A}$ as encoding a large amount of information about its square root $\mathbf{B}$. In particular $\mathbf{a}_{ij} = \mathbf{b}_i^T \mathbf{b}_j$.
- Use this view to find a low-rank approximation to $\mathbf{B}$ using sublinear accesses to $\mathbf{A}$.
- Since $\mathbf{B}$ has the same singular vectors as $\mathbf{A}$ and $\sigma_i(\mathbf{B}) = \sqrt{\sigma_i(\mathbf{A})}$, a low-rank approximation of $\mathbf{B}$ can used to find one for $\mathbf{A}$, albiet with a $\sqrt{n}$ factor loss in quality.
- Obtain near-optimal complexity using ridge leverage scores to sample both rows and columns of $\mathbf{A}$.

OPEN QUESTIONS

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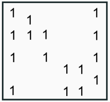
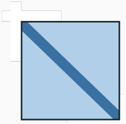

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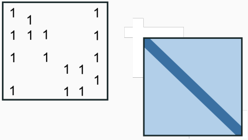

- What can we do when we have PSD matrices with additional structure? E.g. kernel matrices.

Thanks! Questions?