COMPSCI 690RA: Randomized Algorithms and Probabilistic Data Analysis

Prof. Cameron Musco

University of Massachusetts Amherst. Spring 2022.

Lecture 9

Logistics

- Problem Set 3 is due 4/15 at 8pm.
- Project progress report due this Friday, 4/8. Submit a pdf via email. 1-2 pages.
- Weekly quiz due next Tuesday at 8pm.

Summary

Last Week: Random sketching and subspace embedding.

- Subspace embedding via leverage score sampling.
- Analysis via matrix concentration bounds.
- Spectral graph sparsification via leverage score sampling.

$$\begin{bmatrix} S \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} SA \end{bmatrix}$$

Summary

Last Week: Random sketching and subspace embedding.

- Subspace embedding via leverage score sampling.
- Analysis via matrix concentration bounds.
- Spectral graph sparsification via leverage score sampling.

Today:

- Finish spectral graph sparsification and physical interpretation
- Start on Markov chains and their analysis
- Markov chain based algorithms for 2-SAT and 3-SAT.
- Gambler's ruin.

Spectral Graph Sparsification

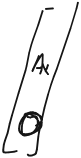
Theorem (Subspace Embedding via Leverage Score Sampling)

For any $A \in \mathbb{R}^{n \times d}$ with left singular vector matrix U, let $\tau_i = \|U_{i,:}\|_2^2$ and $p_i = \frac{\tau_i}{\sum \tau_i}$. Let $\mathbf{S} \in \mathbb{R}^{m \times n}$ have $\mathbf{S}_{:,j}$ independently set to $\frac{1}{\sqrt{mp_i}} \cdot e_i^T$ with probability p_i.

Then, if $m = O\left(\frac{d \log(d/\delta)}{\epsilon^2}\right)$, with probability $\geq 1 - \delta$, $\mathbf{S}$ is an ϵ-subspace embedding for A.

$O\left(\frac{d + \log(1/\delta)}{\varepsilon^2}\right)$

- Matches oblivious random projection up to the $\log d$ factor.
- Variational characterization: $\tau_i = \max_{x \in \mathbb{R}^d} \frac{[Ax](i)^2}{\|Ax\|_2^2}$.



$\lambda_1(A) = \max \frac{x^T A x}{\|x\|_2}$

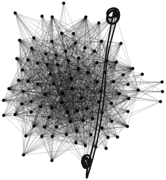
Spectral Graph Sparsification

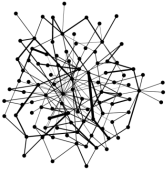
$V = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$V^T L V = \text{cut}.$

cut indicator vector $V = \begin{bmatrix} 1 \\ -1 \\ \vdots \\ 1 \end{bmatrix}$

$V^T L_G V = 4 \cdot \text{cut}$





n nodes

m edges

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

vertex-edge incidence matrix B

$L = B^T B$

- Given a graph G, find a (weighted) subgraph G' with many fewer edges such that: $(1-\epsilon)L_G \preceq L_{G'} \preceq (1+\epsilon)L_G$.
- Equivilantly, letting $B \in \mathbb{R}^{m \times n}$ be the vertex-edge incidence matrix of G, find a sampling matrix $\mathbf{S}$ that is an ϵ-subspace embedding for B. I.e, $\underbrace{B^T \mathbf{S}^T \mathbf{S} B}_{L_{G'}} \approx_\epsilon \underbrace{B^T B}_{L_G}$.

  $\|SBx\|_2 \simeq \|Bx\|_2 \quad \forall x$

$O(n)$

$O(n \log n)$

- Sampling edges according to their leverage scores in B gives an ϵ-spectral sparsifier with just $O(n \log n / \epsilon^2)$ edges.
- Can be used to approximate many properties of G, including the size of all cuts.

Leverage Scores and Effective Resistance

A spectral sparsifier G' of G with $O(n \log n/\epsilon^2)$ edges can be constructed by sampling rows of the vertex-edge incidence matrix (edges) via their leverage scores. What are these leverage scores?

Leverage Scores and Effective Resistance

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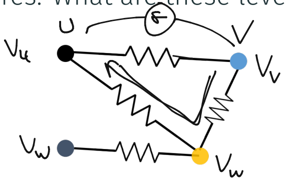


- View each edge as a 1-Ohm resistor.
- If we fix a current of 1 between u, v, the voltage drop across the nodes is known as the effective resistance between u and v.

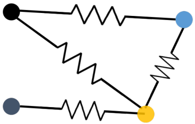
$$V = \overset{1}{I} R$$

resistance

voltage current

Leverage Scores and Effective Resistance

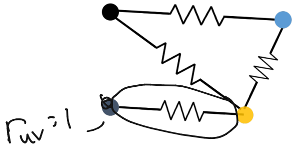
A spectral sparsifier G' of G with $O(n \log n/\epsilon^2)$ edges can be constructed by sampling rows of the vertex-edge incidence matrix via their leverage scores. What are these leverage scores?



- View each edge as a 1-Ohm resistor.
- If we fix a current of 1 between u, v, the voltage drop across the nodes is known as the effective resistance between u and v.
- We will show that the leverage score of each edge is exactly equal to its effective resistance.

Leverage Scores and Effective Resistance

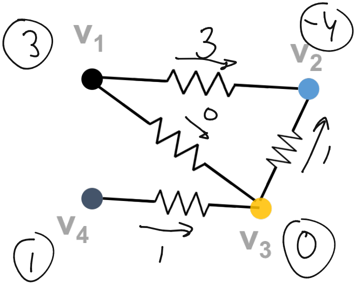
A spectral sparsifier G' of G with $O(n \log n/\epsilon^2)$ edges can be constructed by sampling rows of the vertex-edge incidence matrix via their leverage scores. What are these leverage scores?



- View each edge as a 1-Ohm resistor.
- If we fix a current of 1 between u, v, the voltage drop across the nodes is known as the effective resistance between u and v.
- We will show that the leverage score of each edge is exactly equal to its effective resistance.
- Intuitively, to form a spectral sparsifier, we should sample high resistance edges with high probability, since they are 'bottlenecks'.

Electrical Flows

For a flow $f \in \mathbb{R}^m$, the currents going into each node are given by $B^T f$.



$$
\begin{array}{c|cccc}
 & (1,2) & (1,3) & (2,3) & (3,4) \\
\hline
1 & 1 & 1 & 0 & 0 \\
2 & -1 & 0 & 1 & 0 \\
3 & 0 & -1 & -1 & 1 \\
4 & 0 & 0 & 0 & -1
\end{array}
\quad
\mathbf{B^T}
\quad
\mathbf{f} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ -1 \end{bmatrix}
\quad = \quad
\begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \end{bmatrix}
$$

Electrical Flows

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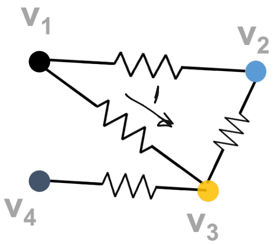


$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{\mathbf{B^T}} \underbrace{\begin{bmatrix} 3 \\ 0 \\ -1 \\ -1 \end{bmatrix}}_{\mathbf{f}} = \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

$$B^T f = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

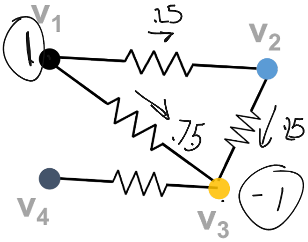
The electrical flow when one unit of current is sent from u to v is:

$$f^e = \underset{f : B^T f = b_{u,v}}{\arg\min} \|f\|_2.$$

Since power (energy/time) is given by $P = I^2 \cdot R$.

Electrical Flows

For a flow $f \in \mathbb{R}^m$, the currents going into each node are given by $B^T f$.



$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{\mathbf{B^T}} \underbrace{\begin{bmatrix} .5 \\ .5 \\ .5 \\ 0 \end{bmatrix}}_{\mathbf{f}} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

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$f(1)^2 + f(2)^2 + f(3) + f(4)^2 = \|f\|_2^2$

Leverage Scores and Effective Resistance

$$f^e = \arg\min_{f: B^T f = b_{u,v}} \|f\|_2.$$

By Ohm's law, the voltage drop across (u, v) (i.e., the effective resistance) is simply the entry $f^e_{u,v}$ (since u, v is a unit resistor).

Leverage Scores and Effective Resistance

$\begin{bmatrix} & B^T & \end{bmatrix} \begin{bmatrix} f \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

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- To solve for f, note that we can assume that f is in the column span of B. Otherwise, it would not have minimal norm. So $f = B\phi$ for some vector $\phi \in \mathbb{R}^n$.

$B^T f = b_{u,v}$

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- Then need to solve $B^T B\phi = b_{u,v}$. I.e., $L\phi = b_{u,v}$. ϕ is unique up to its component in the null-space of L.

Leverage Scores and Effective Resistance

$$f^e = \arg\min_{f: B^T f = b_{u,v}} \|f\|_2.$$

$$\begin{bmatrix} 1 & & -1 & \\ 1 & -1 & & \\ & \vdots & & \\ & \vdots & & \end{bmatrix} \begin{bmatrix} \phi \end{bmatrix} = \begin{bmatrix} f \end{bmatrix} = \begin{bmatrix} \phi(1) - \phi(3) \\ \\ \\ \end{bmatrix}$$

By Ohm's law, the voltage drop across (u, v) (i.e., the effective resistance) is simply the entry $f^e_{u,v}$ (since u, v is a unit resistor).

- To solve for f, note that we can assume that f is in the column span of B. Otherwise, it would not have minimal norm. So $f = B\phi$ for some vector $\phi \in \mathbb{R}^n$. – voltages
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rank(nullspace(L)) = 1

$$\phi = \begin{bmatrix} 1 \\ -1 \\ .5 \end{bmatrix} \quad \phi = \begin{bmatrix} 2 \\ 0 \\ 1.5 \end{bmatrix}$$

L

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Leverage Scores and Effective Resistance

$$f^e = \underset{f: B^T f = b_{u,v}}{\arg\min} \|f\|_2.$$

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- $\underline{\phi = L^+ b_{u,v}}$. $L^{-1} b_{u,v}$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$f^e = \arg\min_{f: B^T f = b_{u,v}} \|f\|_2.$$

$L = V \Lambda V^T$

$L^{-1} = V \Lambda^{-1} V^T$

$L = V \Lambda^{+} V^T$

$\begin{bmatrix} 1/\lambda_1 & & \\ & 1/\lambda_2 & \\ & & 0 \end{bmatrix}$

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- Then need to solve $B^T B \phi = b_{u,v}$. I.e., $L\phi = b_{u,v}$. ϕ is unique up to its component in the null-space of L.
- $\phi = L^+ b_{u,v}$.
- Gives $f^e = BL^+ b_{u,v}$. So $f^e_{u,v}$ is just $b^T_{u,v} L^+ b_{u,v} = b_{u,v}(B^T B)^+ b_{u,v}$.

$\begin{bmatrix} B \\ b_{u,v}^T \end{bmatrix} \begin{bmatrix} L^+ \end{bmatrix} \begin{bmatrix} b_{u,v} \end{bmatrix}$

The effective resistance across edge (u, v) is given by

$$\overset{T}{b_{u,v}}\overset{L^+}{(B^T B)^+} b_{u,v} = e_{u,v}^T B (B^T B)^+ B^T e_{u,v}.$$

$$\overset{\mathbf{b_{u,v}}^{\mathbf{T}}}{\begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}} \; L^+ \; \overset{\mathbf{b_{u,v}}}{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \overset{\mathbf{B}}{\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}} \; L^+ \; \overset{\mathbf{B}^{\mathbf{T}}}{\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The effective resistance across edge (u, v) is given by

$$b_{u,v}(B^TB)^+b_{u,v} = e_{u,v}^T B(B^TB)^+B^T e_{u,v}.$$

$A = U\Sigma V^T$

$A^+ = V\Sigma^{-1}V^T$

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}}_{\mathbf{b_{u,v}^T}} \mathbf{L^+} \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{\mathbf{b_{u,v}}} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}}_{\mathbf{B}} \mathbf{L^+} \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{\mathbf{B^T}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Write $B = U\Sigma V^T$ in its SVD.

$$e_{u,v}^T B\underbrace{(B^TB)^+}B^T e_{u,v} = e_{u,v}^T U\Sigma \overbrace{V^T(V}^{I}\Sigma^{-2}\overbrace{V^T)V}^{I}\Sigma U^T e_{u,v}$$

$B^TB = V\Sigma \cancel{U^TU}\Sigma V^T$

$V\Sigma^2V^T$

$e_{u,v}^T\, U \Sigma \Sigma^{-2} \Sigma\, U^T e_{u,v}$

$e_{u,v}^T\, U U^T e_{u,v}$

Leverage Scores and Effective Resistance

The effective resistance across edge (u, v) is given by

$$b_{u,v}(B^TB)^+b_{u,v} = e_{u,v}^TB(B^TB)^+B^Te_{u,v}.$$

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}}_{\mathbf{b_{u,v}}^{\mathbf{T}}} \; \mathbf{L^+} \; \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{\mathbf{b_{u,v}}} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}}_{\mathbf{B}} \; \mathbf{L^+} \; \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{\mathbf{B^T}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Write $B = U\Sigma V^T$ in its SVD.

$$\begin{aligned} e_{u,v}^TB(B^TB)^+B^Te_{u,v} &= e_{u,v}^TU\Sigma V^T(V\Sigma^{-2}V^T)V\Sigma U^Te_{u,v} \\ &= e_{u,v}^TUU^Te_{u,v} \end{aligned}$$

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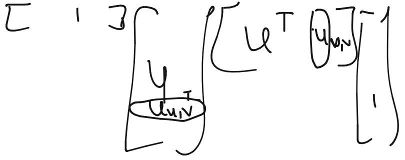

leverage score.

Leverage Scores and Effective Resistance

$B^T D^{-1} D B$

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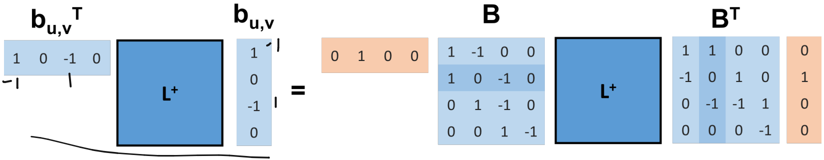

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I.e., the effective resistance is exactly the leverage score of the corresponding row in B.

Markov Chains

Markov Chain Definition

- A discrete time stochastic process is a collection of random variables $\underbrace{X_0, X_1, X_2, \ldots,}$
- A discrete time stochastic process is a Markov chain if is it memoryless:

$$\Pr(\underline{X_t = a_t}|\underline{X_{t-1} = a_{t-1}, \ldots, X_0 = a_0}) = \Pr(X_t = a_t|\underline{X_{t-1} = a_{t-1}})$$
$$= \underline{P_{a_{t-1},a_t}}.$$

Think-Pair-Share: In a Markov chain, is X_t independent of $X_{t-2}, X_{t-3}, \ldots, X_0$? $\boxed{\text{NO}}$

$\to X_i = \pm 1$ w.p $1/2$ $\quad X_i$ is independent of all j, $j \neq i$

$\to X_i$ = # number of heads seen up to step i

Transition Matrix

A Markov chain $X_0, X_1, \ldots$ where each X_i can take m possible values, is specified by the transition matrix $P \in [0,1]^{m \times m}$ with

$$P_{j,k} = \Pr(X_{i+1} = k | X_i = j).$$

P

$$\begin{array}{c|cccc} 1 & .5 & .5 & 0 & 0 \\ 2 & .25 & .25 & .5 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ 4 & 0 & .5 & .5 & 0 \end{array}$$

Transition Matrix

A Markov chain $\mathbf{X}_0, \mathbf{X}_1, \ldots$ where each $\mathbf{X}_i$ can take m possible values, is specified by the transition matrix $P \in [0,1]^{m \times m}$ with

$$P_{j,k} = \Pr(\mathbf{X}_{i+1} = k | \mathbf{X}_i = j).$$

Let $q_i \in [0,1]^{1 \times m}$ be the distribution of $\mathbf{X}_i$. Then $q_{i+1} = q_i P$.

$[.5 \ .5 \ 0 \ 0]$

P

.5	.5	0	0
.25	.25	.5	0
0	1	0	0
0	.5	.5	0

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Let $q_i \in [0,1]^{1 \times m}$ be the distribution of X_i. Then $q_{i+1} = q_i P$.

$X_0 = 1$

$Pr(X_1 = 1 | X_0 = 1) = .5$

$Pr(X_1 = 2 | X_0 = 1) = .5$

$\mathbf{q_0}$

1	0	0	0

$\mathbf{P}$

.5	.5	0	0
.25	.25	.5	0
0	1	0	0
0	.5	.5	0

=

$\mathbf{q_1}$

.5	.5	0	0

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$$P_{j,k} = \Pr(\mathbf{X}_{i+1} = k | \mathbf{X}_i = j).$$

Let $q_i \in [0,1]^{1 \times m}$ be the distribution of $\mathbf{X}_i$. Then $q_{i+1} = q_i P$.

$$\underset{\mathbf{q_1}}{\begin{bmatrix} \underline{.5} & \underline{.5} & 0 & 0 \end{bmatrix}} \underset{\mathbf{P}}{\begin{bmatrix} \underline{.5} & .5 & 0 & 0 \\ \underline{.25} & .25 & .5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & .5 & .5 & 0 \end{bmatrix}} = \underset{\mathbf{q_2}}{\begin{bmatrix} .375 & .375 & .25 & 0 \end{bmatrix}}$$

$Pr(X_2 = 1)$

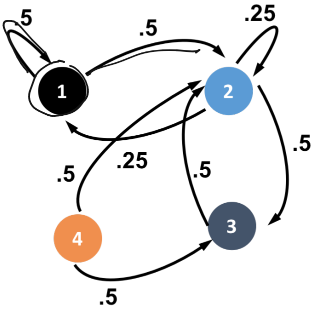
$\sum_{j=1,2,3,4} Pr(X_1 = j) \cdot Pr(X_2 = 1 | X_1 = j)$

$.5 \cdot .5 + .5 \cdot .25$

$= .375$

Graph View

Often viewed as an underlying state transition graph. Nodes correspond to possible values that each X_i can take.



P

.5	.5	0	0
.25	.25	.5	0
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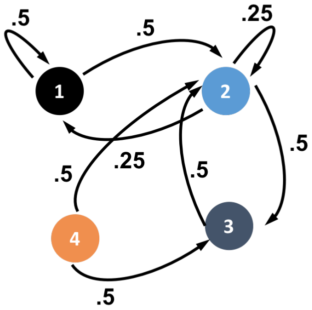
Graph View

Often viewed as an underlying state transition graph. Nodes correspond to possible values that each X_i can take.



P

$$\begin{bmatrix} .5 & .5 & 0 & 0 \\ .25 & .25 & .5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & .5 & .5 & 0 \end{bmatrix}$$

The Markov chain is irreducible if the underlying graph consists of single strongly connected component.

2-SAT

Motivating Example: Find a satisfying assignment for a 2-CNF formula with *n* variables. $x_1 = 0$ $x_2 = 1$ $x_3 = 1$, $x_4 = 1$

$$(x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_3) \wedge (x_1 \vee x_2) \wedge (x_4 \vee \bar{x}_3) \wedge (x_4 \vee \bar{x}_1)$$

solvable in poly tile.

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A simple 'local search' algorithm: $n = \#$ vars.

1. Start with an arbitrary assignment.
2. Repeat $2mn^2$ times, terminating if a satisfying assignment is found:
   - Chose an arbitrary unsatisfied clause.
   - Pick one of the variables in the clause uniformly at random, and switch the assignment of the variable.
3. If a valid assignment is not found, return that the formula is unsatisfiable.

even to define X_i, I need the formula to be satisfiable.

2-SAT

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$$(x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_3) \wedge (x_1 \vee x_2) \wedge (x_4 \vee \bar{x}_3) \wedge (x_4 \vee \bar{x}_1)$$

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Claim: If the formula is satisfiable, the algorithm finds a satisfying assignment with probability $\geq 1 - 2^{-m}$.

Randomized 2-SAT Analysis

Fix a satisfying assignment S. Let $X_i \leq n$ be the number of variables that are assigned the same values as in S, at step i.

S

1	0	1	0	0	0	1	1

$S_i = 5$

assignment i

1	**0**	0	1	1	**0**	**1**	**1**

- $X_{i+1} = X_i \pm 1$ since we flip one variable in an unsatisfied clause.

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$X_0, X_1, X_2, \ldots$

- $\mathsf{X}_{i+1} = \mathsf{X}_i \pm 1$ since we flip one variable in an unsatisfied clause.
- $\Pr(\underline{\mathsf{X}_{i+1}} = \underline{\mathsf{X}_i + 1}) \geq 1/2$
- $\Pr(\underline{\mathsf{X}_{i+1} = \mathsf{X}_i} - 1) \leq 1/2$

$x_1 = 1, \ x_2 = 1$

$$(x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_3) \wedge (\underline{\overset{0}{x_1} \vee \overset{1}{x_2}}) \wedge (x_4 \vee \bar{x}_3) \wedge (x_4 \vee \bar{x}_1)$$

Coupling to a Markov Chain

The number of correctly assigned variables at step i, X_i, obeys

$$\Pr(X_{i+1} = X_i + 1) \geq \frac{1}{2} \quad \text{and} \quad \Pr(X_{i+1} = X_i - 1) \leq \frac{1}{2}.$$

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Is $X_0, X_1, X_2, \ldots$ a Markov chain? $\boxed{\text{No}}$

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Is $X_0, X_1, X_2, \ldots$ a Markov chain?

Define a Markov chain $Y_0, Y_1, \ldots$ such that $Y_0 = X_0$ and:

$$\begin{aligned}
\Pr(Y_{i+1} = 1 | Y_i = 0) &= 1 \\
\Pr(Y_{i+1} = j + 1 | Y_i = j) &= 1/2 \text{ for } 1 \leq j \leq n-1 \\
\Pr(Y_{i+1} = j - 1 | Y_i = j) &= 1/2 \text{ for } 1 \leq j \leq n-1 \\
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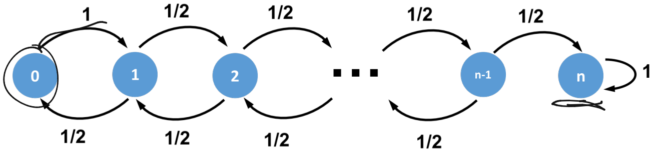
\end{aligned}$$

- Our algorithm terminates as soon as $X_i = n$. We expect to reach this point only more slowly with Y_i. So it suffices to argue that $Y_i = n$ with high probability for large enough i.
- Formally could use a coupling argument (see Chapter 11 of Mitzenmacher Upfal.)

Simple Markov Chain Analysis

Want to bound the expected time required to have $\mathbf{Y}_i = n$.



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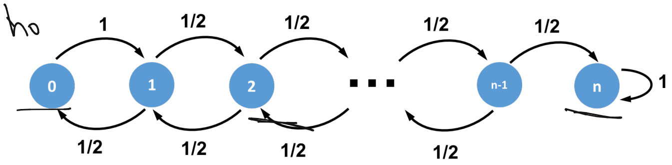

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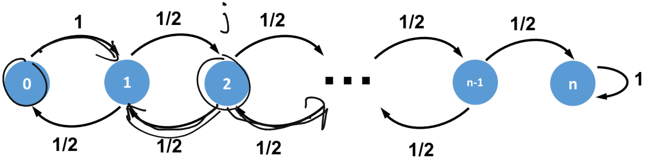

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$h_{j-1} = h_j + 2(j-1) + 1$

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So in total we have:

$$h_0 = h_1 + 1 = h_2 + 3 + 1 = \ldots = \sum_{j=0}^{n-1}(2j+1) \qquad .$$

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Upshot: Consider the Markov chain $Y_0, Y_1, \ldots$, and let i^* be the minimum i such $Y_{i^*} = n$. Then $\mathbb{E}[i^*] \leq n^2$.

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bad question

~~**Check-in Question:** For a fixed i, what roughly is $\mathbb{E}[Y_i]$?~~

random ± 1 walk on an infinite line

$\mathbb{E}\, Y_i = 0$

$\mathbb{E}\, Y_i^2 = i$

$\mathbb{E}|Y_i| \approx O(\sqrt{i})$

3-SAT

More Challenging Problem: Find a satisfying assignment for a 3-CNF formula with n variables.

$$(x_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee x_4) \wedge (x_1 \vee x_2 \vee \bar{x}_3).$$

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- Will see that our simple Markov chain approach gives an $O(1.3334^n)$ time algorithm.

$O(\sqrt{n} \cdot (\frac{4}{3})^n)$

$2^{40} = 1000^4 = 1$ tril

$\left(\frac{4}{3}\right)^{40} \approx 100{,}000$

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- Note that the exponential time hypothesis conjectures that $O(c^n)$ is needed to solve 3-SAT for some constant $c > 1$. The strong exponential time hypothesis conjectures that for $k \to \infty$, solving k-SAT requires $O(2^n)$ time.

Randomized 3-SAT Algorithm

1. Start with an arbitrary assignment.
2. Repeat *m* times, terminating if a satisfying assignment is found:
    - Chose an arbitrary unsatisfied clause.
    - Pick one of the variables in the clause uniformly at random, and switch the assignment of the variable.
3. If a valid assignment is not found, return that the formula is unsatisfiable.

Randomized 3-SAT Analysis

As in the 2-SAT setting, let X_i be the number of correctly assigned variables at step i. We have:

current: 0 0 0

$$(x_1 \vee x_2 \vee x_3)$$

"correct" assignment: 1 0 0

$$\Pr(X_i = X_{i-1} + 1) \geq 1/3$$

$$\Pr(X_i = X_{i-1} - 1) \leq 2/3$$

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$$\Pr(X_i = X_{i-1} + 1) \geq$$
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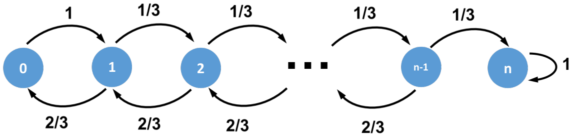
Define the coupled Markov chain $Y_0, Y_1, \ldots$ as before, but with $Y_i = Y_{i-1} + 1$ with probability $1/3$ and $Y_i = Y_{i-1} - 1 = 2/3$.



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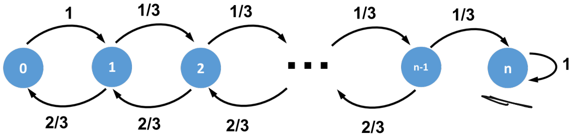

How many steps do you expect are needed to reach $Y_i = n$?

$O(2^n)$

Randomized 3-SAT Analysis

Letting h_j be the expected number of steps to reach n when starting at node j,

$$h_n = 0$$
$$h_0 = h_1 + 1$$
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- Thus, in expectation, our algorithm takes at most $\approx 2^{n+2}$ steps to find a satisfying assignment if there is one.

Take a random walk on an infinite line

$\mathbb{E}\, Y_i = -\frac{i}{3}$ $\quad \Pr(Y_i \geq n) \leq \Pr(|Y_i - \mathbb{E}Y_i| \geq n + \frac{i}{3}) = \exp(-(n+\frac{i}{3}))$

$\exp(-n)$

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- Is this an interesting result?

NO worse than 2^n

Modified 3-SAT Algorithm

Key Idea: If we pick our initial assignment uniformly at random, we will have $\mathbb{E}[X_0] = n/2$. With very small, but still non-negligible probability, X_0 will be much larger, and our random walk will be more likely to find a satisfying assignment.

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Modified Randomized 3-SAT Algorithm:

Repeat m times, terminating if a satisfying assignment is found:

1. Pick a uniform random assignment for the variables.
2. Repeat $3n$ times, terminating if a satisfying assignment is found:
    - Chose an arbitrary unsatisfied clause.
    - Pick one of the variables in the clause uniformly at random, and switch the assignment of the variable.

If a valid assignment is not found, return that the formula is unsatisfiable.

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Consider a single random assignment with $X_0 = n - j$. I.e., we need to correct j variables to find a satisfying assignment.

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Let q_j be a lower bound on the success probability in this case. Since $j \le n$ and since we run the search process for $3n$ steps,

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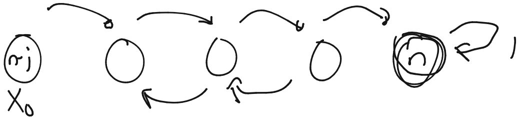

If $X_{3j} = n$ then X_k for $k > 3j$ has $X_k = n$

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Via Stirling's approximation, $\binom{3j}{j} \geq \frac{1}{\sqrt{j}} \cdot \frac{3^{3j-2}}{2^{2j-2}}$, giving:

(wikipedia)

$$
q_j \geq \frac{2^2}{3^2\sqrt{j}} \cdot \frac{\cancel{3^{3j}}}{2^{2j}} \cdot \frac{2^j}{\cancel{3^{3j}}} \approx \frac{1}{\sqrt{j} \cdot 2^j} \geq \underline{\frac{1}{\sqrt{n} \cdot 2^j}}.
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$\left(1+\frac{1}{2}\right)^n$

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Our overall probability of success in a single trial is then lower bounded by:

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&\geq \frac{1}{\sqrt{n} \cdot 2^n} \sum_{j=0}^{n} \binom{n}{j} \cdot \frac{1}{2^j} \qquad \text{"binomial theorem"} \\
&= \frac{1}{\sqrt{n} \cdot 2^n} \cdot \left(\frac{3}{2}\right)^n \lesssim \frac{1}{\sqrt{n}} \cdot \left(\frac{3}{4}\right)^n .
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Thus, if we repeat for $m = O\left(\sqrt{n} \cdot \left(\frac{4}{3}\right)^n\right) = O(\underline{1.33334}^n)$ trials, with very high probability, we will find a satisfying assignment if there is one.

Gambler’s Ruin

Gambler's Ruin

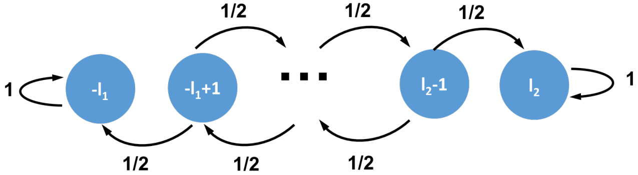
- You and 'a friend' repeatedly toss a fair coin. If it hits heads, you give your friend \$1. If it hits tails, they give you \$1.
- You start with $\$\ell_1$ and your friend starts with $\$\ell_2$. When either of you runs out of money the game terminates.
- What is the probability that you win $\$\ell_2$?

Gambler's Ruin Markov Chain

Let $X_0, X_1, \ldots$ be the Markov chain where X_i is your profit at step i. $X_0 = 0$ and:

$$P_{-\ell_1,-\ell_1} = P_{\ell_2,\ell_2} = 1$$

$$P_{j,j+1} = P_{j,j-1} = 1/2 \text{ for } -\ell_1 < j < \ell_2$$



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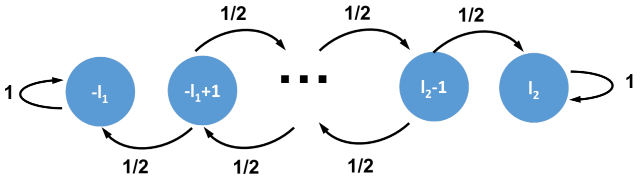

- ℓ_1 and ℓ_2 are absorbing states.
- All j with $-\ell_1 < j < \ell_2$ are transient states. I.e., $\Pr[X_{i'} = j \text{ for some } i' > i \,|\, X_i = j] < 1$.

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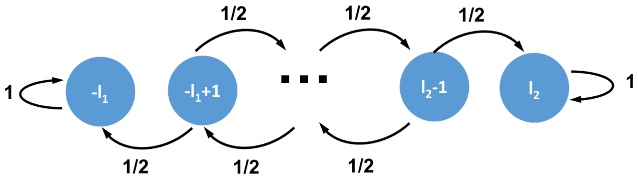

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Observe that this Markov chain is also a Martingale since $\mathbb{E}[X_{i+1}|X_i] = X_i$.

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Solving for q, we have $q = \frac{\ell_1}{\ell_1+\ell_2}$.

Gambler's Ruin Thought Exercise

What if you always walk away as soon as you win just \$1. Then what is your probability of winning, and what are your expected winnings?