# COMPSCI 690RA: Randomized Algorithms and Probabilistic Data Analysis

Prof. Cameron Musco

University of Massachusetts Amherst. Spring 2022.

Lecture 7

# Logistics

- I'll return midterms at the end of class.
- Overall the class did very well – mean was a 29.75 out of 36 ($\approx 83\%$).
- If you are not happy with your performance, message me and we can chat about it. I'm also happy to review solutions in office hours.
- I plan to release Problem Set 3 by end of this week.
- 1 page progress report on Final Project due 4/8.
- Weekly quiz due next Tuesday at 8pm.

# Summary

Randomized Linear Algebra Before Break:

- Freivald's algorithm for matrix product testing.
- Hutchinson's method for trace estimation. Analysis via linearity of variance for pairwise-independent random variables.
- Approximate matrix multiplication via norm-based sampling. Analysis via outer-product view of matrix multiplication.
- Application to fast randomized low-rank approximation.
- Related ideas for sampling for initializing $k$-means clustering – the $k$-means++ algorithm.

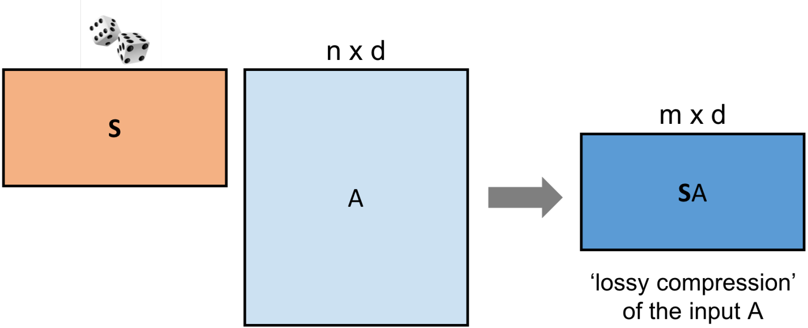
**Today:** Random sketching and the Johnson-Lindenstrauss lemma.

- Subspace embedding and $\epsilon$-net arguments.
- Application to fast over-constrained linear regression.

# Linear Sketching

# Linear Sketching

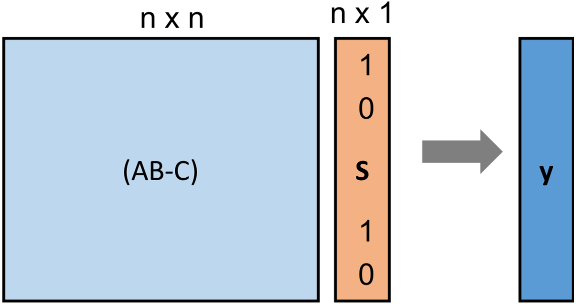
Given a large matrix $A \in \mathbb{R}^{n \times d}$, we pick a random linear transformation $\mathbf{S} \in \mathbb{R}^{m \times n}$ and compute $\mathbf{S}A$ (alternatively, pick $\mathbf{S} \in \mathbb{R}^{d \times m}$ and compute $A\mathbf{S}$). Using $\mathbf{S}A$ we can approximate many computations involving $A$.




What algorithms have we seen in class that are based on linear sketching?

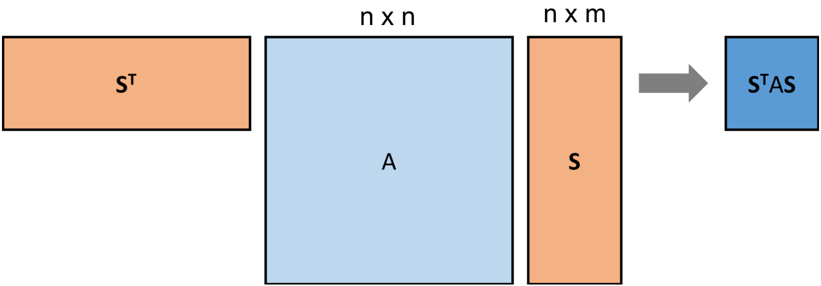
# Linear Sketching Examples

Freivald's Algorithm:



# Linear Sketching Examples

Hutchinson's Trace Estimator:



# Linear Sketching Examples

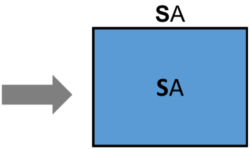
## Graph Connectivity via $\ell_0$ sampling:

$l_0$ sampling matrix **S**

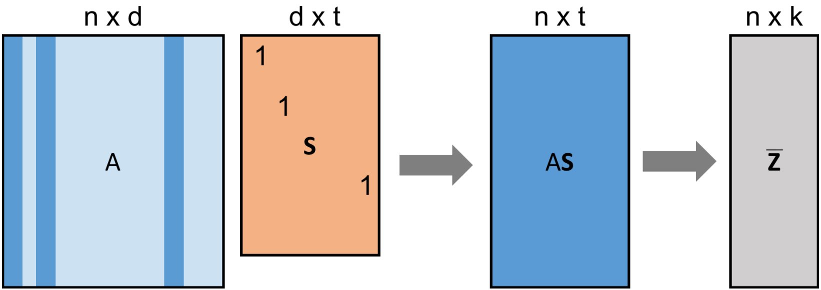
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | -1 | 0 | 0 | 1 | -1 | 0 | 1 |
| -1 | 0 | 1 | 1 | 0 | 0 | -1 | 0 |
| 1 | 1 | -1 | 0 | -1 | -1 | 0 | 1 |
| 0 | -1 | -1 | -1 | 1 | 1 | 1 | 0 |

| $v_1$ | $v_2$ | $v_3$ | $v_4$ |
|---|---|---|---|
| 1 | -1 | 0 | 0 |
| 0 | 1 | 0 | -1 |
| 0 | 0 | 1 | -1 |
| -1 | 0 | 1 | 0 |
| 1 | 0 | -1 | 0 |
| 0 | 1 | -1 | 0 |
| 1 | 0 | 0 | -1 |
| 0 | 0 | 1 | -1 |

vertex-edge
incidence matrix A



Norm-Based Sampling for AMM/Low-Rank Approximation:



# Subspace Embedding

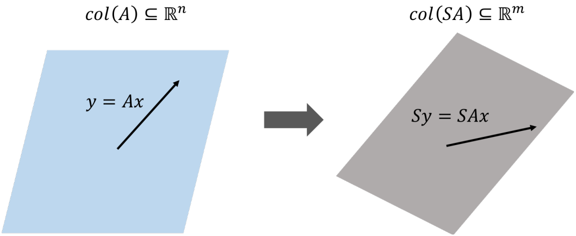
It is helpful to define general guarantees for sketches, that are useful in many problems.

**Definition (Subspace Embedding)**

$S \in \mathbb{R}^{m \times d}$ is an $\epsilon$-subspace embedding for $A \in \mathbb{R}^{n \times d}$ if, for all $x \in \mathbb{R}^d$,

$$(1-\epsilon)\|Ax\|_2 \leq \|SAx\|_2 \leq (1+\epsilon)\|Ax\|_2.$$

I.e., $S$ preserves the norm of any vector $Ax$ in the column span of $A$.



# Subspace Embedding Application

## Theorem (Sketched Linear Regression)

*Consider $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$. We seek to find an approximate solution to the linear regression problem:*

$$\arg\min_{x \in \mathbb{R}^d} \|Ax - b\|_2.$$

*Let $S \in \mathbb{R}^{m \times d}$ be an $\epsilon$-subspace embedding for $[A; b] \in \mathbb{R}^{n \times d+1}$. Let $\tilde{x} = \arg\min_{x \in \mathbb{R}^d} \|SAx - Sb\|_2$. Then we have:*

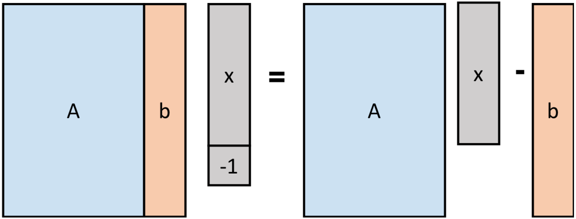
$$\|A\tilde{x} - b\|_2 \leq \frac{1+\epsilon}{1-\epsilon} \cdot \min_{x \in \mathbb{R}^d} \|Ax - b\|_2.$$

- Time to compute $x^* = \arg\min_{x \in \mathbb{R}^d} \|Ax - b\|_2$ is $O(nd^2)$.
- Time to compute $\tilde{x}$ is just $O(md^2)$. For large $n$ (i.e., a highly over-constrained problem) can set $m \ll n$.

# Sketched Regression Proof

**Claim:** Since $S$ is a subspace embedding for $[A; b]$, for all $x \in \mathbb{R}^d$,

$$(1-\epsilon)\|Ax - b\|_2 \leq \|SAx - Sb\|_2 \leq (1+\epsilon)\|Ax - b\|_2.$$



# Sketched Regression Proof

**Claim:** Since $S$ is a subspace embedding for $[A; b]$, for all $x \in \mathbb{R}^d$,

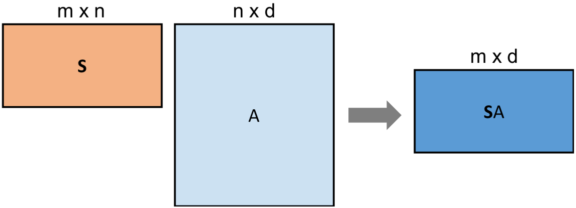
$$(1-\epsilon)\|Ax - b\|_2 \leq \|SAx - Sb\|_2 \leq (1+\epsilon)\|Ax - b\|_2.$$

Let $x^* = \arg\min_{x \in \mathbb{R}^d} \|Ax - b\|_2$ and $\tilde{x} = \arg\min_{x \in \mathbb{R}^d} \|SAx - Sb\|_2$. We have:

$$\begin{aligned}\|A\tilde{x} - b\|_2 \leq \frac{1}{1-\epsilon}\|SAx - Sb\|_2 &\leq \frac{1}{1-\epsilon} \cdot \|SAx^* - Sb\|_2 \\ &\leq \frac{1+\epsilon}{1-\epsilon} \cdot \|Ax^* - b\|_2.\end{aligned}$$

# Subspace Embedding Intuition

**Think-Pair-Share 1:** Assume that $n > d$ and that $\text{rank}(A) = d$. If $S \in \mathbb{R}^{m \times n}$ an is an $\epsilon$-subspace embedding for $A$ with $\epsilon < 1$, how large must $m$ be? **Hint:** Think about $\text{rank}(SA)$ and/or the nullspace of $SA$.




**Think-Pair-Share 2:** Describe how to deterministically compute a subspace embedding $S$ with $m = d$ and $\epsilon = 0$ in $O(nd^2)$ time.

# Optimal Subspace Embedding

Let $Q \in \mathbb{R}^{n \times d}$ be an orthonormal basis for the columns of $A$. Then any vector $Ax$ in $A$'s column span can be written as $Qy$ for some $y \in \mathbb{R}^d$.

Let $S = Q^T$. $S \in \mathbb{R}^{d \times n}$ (i.e., $m = d$) and further, for any $x \in \mathbb{R}^d$

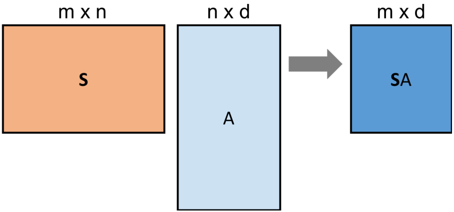
$$\|SAx\|_2^2 = \|Q^T Qy\|_2^2 = \|y\|_2^2 = \|Ax\|_2^2.$$

How would you compute $Q$?

# Randomized Subspace Embedding

## Theorem (Oblivious Subspace Embedding)

*Let* $\mathbf{S} \in \mathbb{R}^{m \times d}$ *be a random matrix with i.i.d.* $\pm 1/\sqrt{m}$ *entries. Then if* $m = O\left(\frac{d + \log(1/\delta)}{\epsilon^2}\right)$*, for any* $A \in \mathbb{R}^{n \times d}$*, with probability* $\geq 1 - \delta$*,* $\mathbf{S}$ *is an* $\epsilon$*-subspace embedding of* $A$*.*




- $\mathbf{S}$ can be computed without any knowledge of $A$.
- Still achieves near optimal compression.
- Constructions where $\mathbf{S}$ is sparse or structured, allow efficient computation of $\mathbf{S}A$ (fast JL-transform, input-sparsity time algorithms)

# Oblivious Subspace Embedding Proof

# Proof Outline

1. **Distributional Johnson-Lindenstrauss**: For $\mathbf{S} \in \mathbb{R}^{m \times d}$ with i.i.d. $\pm 1/\sqrt{m}$ entries, for any fixed $y \in \mathbb{R}^n$, with probability $1-\delta$ for very small $\delta$, $(1-\epsilon)\|y\|_2 \leq \|\mathbf{S}y\|_2 \leq (1+\epsilon)\|y\|_2$.
2. Via a union bound, have that for any fixed set of vectors $\mathcal{N} \subset \mathbb{R}^n$, with probability $1 - |\mathcal{N}| \cdot \delta$, $\|\mathbf{S}y\|_2 \approx_\epsilon \|y\|_2$ for all $y \in \mathcal{N}$.
3. But we want $\|\mathbf{S}y\|_2 \approx_\epsilon \|y\|_2$ for all $y = Ax$ with $x \in \mathbb{R}^d$. This is a linear subspace, i.e., an infinite set of vectors!
4. 'Discretize' this subspace by rounding to a finite set of vectors $\mathcal{N}$, called an $\epsilon$-net for the subspace. Then apply union bound to this finite set, and show that the discretization does not introduce too much error.

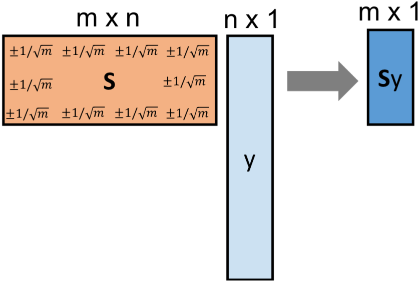
**Remark:** $\epsilon$-nets are a key proof technique in theoretical computer science, learning theory (generalization bounds), random matrix theory, and beyond. They are a key take-away from this lecture.

# Step 1: Distributional JL Lemma

**Theorem (Distributional JL)**

*Let* $\mathbf{S} \in \mathbb{R}^{m \times d}$ *be a random matrix with i.i.d.* $\pm 1/\sqrt{m}$ *entries. Then if* $m = O(\log(1/\delta)/\epsilon^2)$, *for any fixed* $y \in \mathbb{R}^n$, *with probability* $\geq 1 - \delta$, $(1-\epsilon)\|y\|_2 \leq \|\mathbf{S}y\|_2 \leq (1+\epsilon)\|y\|_2$.

I.e., via a random matrix, we can compress any vector from $n$ to $\approx \log(1/\delta)/\epsilon^2$ dimensions, and approximately preserve its norm. A bit surprising maybe that $m$ does not depend on $n$ at all.

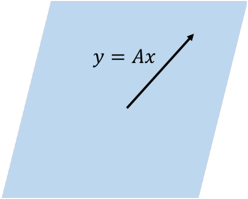


# Restriction to Unit Ball

Want to show that with high probability, $\|\mathbf{S}y\|_2 \approx_\epsilon \|y\|_2$ for all $y \in \{Ax : x \in \mathbb{R}^d\}$. I.e., for all $y \in \mathcal{V}$, where $\mathcal{V}$ is $A$'s column span.

**Observation:** Suffices to prove $\|\mathbf{S}y\|_2 \approx_\epsilon \|y\|_2 = 1$ for all $y \in S_{\mathcal{V}}$ where

$$S_{\mathcal{V}} = \{y \,:\, y \in \mathcal{V} \text{ and } \|y\|_2 = 1\}.$$




**Proof:** For any $y \in \mathcal{V}$, can write $y = \|y\|_2 \cdot \bar{y}$ where $\bar{y} = y/\|y\|_2 \in S_{\mathcal{V}}$.

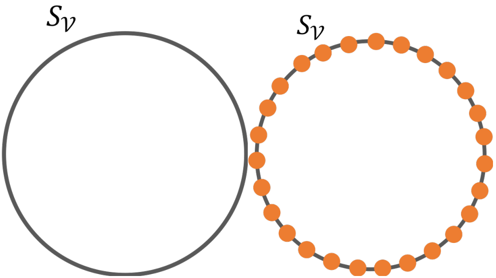
$$(1-\epsilon) \leq \|\mathbf{S}\bar{y}\|_2 \leq (1+\epsilon) \implies$$
$$(1-\epsilon)\cdot\|y\|_2 \leq \|\mathbf{S}\bar{y}\|_2 \cdot \|y\|_2 \leq (1+\epsilon)\cdot\|y\|_2 \implies$$
$$(1-\epsilon)\|y\|_2 \leq \|\mathbf{S}y\|_2 \leq (1+\epsilon)\|y\|_2.$$

# Discretization of Unit Ball

**Theorem**

*For any $\epsilon \leq 1$, there exists a set of points $\mathcal{N}_\epsilon \subset S_\mathcal{V}$ with $|\mathcal{N}_\epsilon| = \left(\frac{4}{\epsilon}\right)^d$ such that, for all $y \in S_\mathcal{V}$,*

$$\min_{w \in \mathcal{N}_\epsilon} \|y - w\|_2 \leq \epsilon.$$




By the distributional JL lemma, if we set $\delta' = \delta \cdot \left(\frac{\epsilon}{4}\right)^d$ then, via a union bound, with probability at least $1 - \delta' \cdot |\mathcal{N}_\epsilon| = 1 - \delta$, for

# Proof Via $\epsilon$-net

**So Far:** If we set $m = \tilde{O}(d/\epsilon^2)$ and pick random $\mathbf{S} \in \mathbb{R}^{m \times n}$, then with probability $\geq 1 - \delta$, $\|\mathbf{S}w\|_2 \approx_\epsilon \|w\|_2$ for all $w \in \mathcal{N}_\epsilon$.

**Expansion via net vectors:** For any $y \in \mathcal{S}_\mathcal{V}$, we can write:

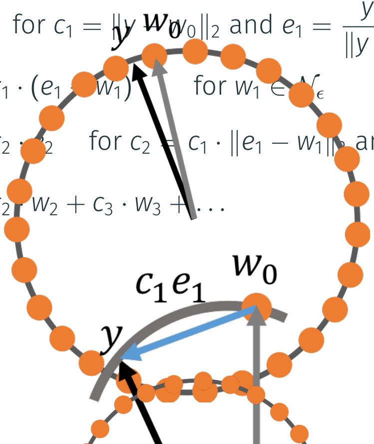
$$y = w_0 + (y - w_0) \qquad \text{for } w_0 \in \mathcal{N}_\epsilon$$

$$= w_0 + c_1 \cdot e_1 \qquad \text{for } c_1 = \|y - w_0\|_2 \text{ and } e_1 = \frac{y - w_0}{\|y - w_0\|_2} \in S_\mathcal{V}$$

$$= w_0 + c_1 \cdot w_1 + c_1 \cdot (e_1 - w_1) \qquad \text{for } w_1 \in \mathcal{N}_\epsilon$$

$$= w_0 + c_1 \cdot w_1 + c_2 \cdot e_2 \qquad \text{for } c_2 = c_1 \cdot \|e_1 - w_1\|_2 \text{ and } e_2 = \frac{e_1 - w_1}{\|e_1 - w_1\|_2} \in S_\mathcal{V}$$

$$= w_0 + c_1 \cdot w_1 + c_2 \cdot w_2 + c_3 \cdot w_3 + \ldots$$



# Proof Via $\epsilon$-net

Have written $y \in S_{\mathcal{V}}$ as $y = w_0 + c_1 w_1 + c_2 w_2 + \ldots$ where $w_0, w_1, \ldots \in \mathcal{N}_\epsilon$, and $c_i \leq \epsilon^i$. By triangle inequality:

$$
\begin{aligned}
\|\mathbf{S}y\|_2 &= \|\mathbf{S}w_0 + c_1\mathbf{S}w_1 + c_2\mathbf{S}w_2 + \ldots\|_2 \\
&\leq \|\mathbf{S}w_0\|_2 + c_1\|\mathbf{S}w_1\|_2 + c_2\|\mathbf{S}w_2\|_2 + \ldots \\
&\leq (1+\epsilon) + \epsilon(1+\epsilon) + \epsilon^2(1+\epsilon) + \ldots \\
&\text{(since via the union bound, } \|\mathbf{S}w\|_2 \approx \|w\|_2 \text{ for all } w \in \mathcal{N}_\epsilon) \\
&\leq \frac{1+\epsilon}{1-\epsilon} \approx 1 + 2\epsilon
\end{aligned}
$$

Similarly, can prove that $\|\mathbf{S}y\|_2 \geq 1 - 2\epsilon$, giving, for all $y \in S_{\mathcal{V}}$ (and hence all $y \in \mathcal{V}$):

$$(1-2\epsilon)\|y\|_2 \leq \|\mathbf{S}y\|_2 \leq (1+2\epsilon)\|y\|_2.$$

# Full Argument

- There exists an $\epsilon$-net $\mathcal{N}_\epsilon$ over the unit ball in $A$'s column span, $S_\mathcal{V}$ with $|\mathcal{N}_\epsilon| \leq \left(\frac{4}{\epsilon}\right)^d$.
- By distributional JL, for $m = O\left(\frac{d \log(1/\epsilon) + \log(1/\delta)}{\epsilon^2}\right)$, with probability $\geq 1 - \delta$, for all $w \in \mathcal{N}_\epsilon$, $\|\mathbf{S}w\|_2 \approx_\epsilon \|w\|_2$.

$\implies$ for all $y \in \mathcal{S}_\mathcal{V}$, $\|\mathbf{S}y\|_2 \approx_\epsilon \|y\|_2$.

$\implies$ for all $y \in \mathcal{V}$, i.e., for all $y = Ax$ for $x \in \mathbb{R}^d$, $\|\mathbf{S}y\|_2 \approx_\epsilon \|y\|_2$.

$\implies$ $\mathbf{S} \in \mathbb{R}^{m \times n}$ is an $\epsilon$-subspace embedding for $A$.

# Net Construction

**Theorem ($\epsilon$-net over $\ell_2$ ball)**

*For any $\epsilon \leq 1$, there exists a set of points $\mathcal{N}_\epsilon \subset S_\mathcal{V}$ with $|\mathcal{N}_\epsilon| = \left(\frac{4}{\epsilon}\right)^d$ such that, for all $y \in S_\mathcal{V}$,*

$$\min_{w \in \mathcal{N}_\epsilon} \|y - w\|_2 \leq \epsilon.$$

Theoretical algorithm for constructing $\mathcal{N}_\epsilon$:

- Initialize $\mathcal{N}_\epsilon = \{\}$.
- While there exists $v \in S_\mathcal{V}$ where $\min_{w \in \mathcal{N}_\epsilon} \|v - w\|_2 > \epsilon$, pick an arbitrary such $v$ and let $\mathcal{N}_\epsilon := \mathcal{N}_\epsilon \cup \{v\}$.

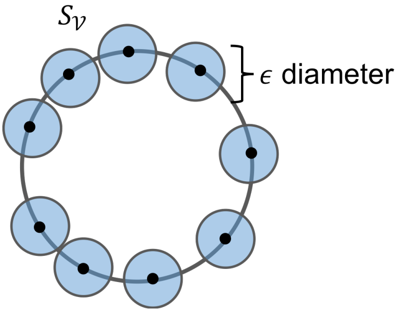
If the algorithm terminates in $T$ steps, we have $|\mathcal{N}_\epsilon| \leq T$ and $\mathcal{N}_\epsilon$ is a valid $\epsilon$-net.

# Net Construction

How large is the net constructed by our theoretical algorithm?

Consider $w, w' \in \mathcal{N}_\epsilon$. We must have $\|w - w'\|_2 > \epsilon$, or we would have not added both to the net.

Thus, we can place an $\epsilon/2$ radius ball around each $w \in \mathcal{N}_\epsilon$, and none of these balls will intersect.




Note that all these balls lie within the ball of radius $(1 + \epsilon/2)$.

# Volume Argument

We have $|\mathcal{N}_\epsilon|$ disjoint balls with radius $\epsilon/2$, lying within a ball of radius $(1+\epsilon/2)$.

In $d$ dimensions, the radius $r$ ball has volume $c_d \cdot r^d$, where $c_d$ is a constant that depends on $d$ but not $r$.

Thus, the total number of balls is upper bounded by:

$$|\mathcal{N}_\epsilon| \leq \frac{(1+\epsilon/2)^d}{(\epsilon/2)^d} \leq \left(\frac{4}{\epsilon}\right)^d.$$

**Remark:** We never actually construct an $\epsilon$-net. We just use the fact that one exists (the output of this theoretical algorithm) in our subspace embedding proof.

# Distributional JL Lemma Proof

# Proofs of Distributional JL Lemma

There are many proofs of the distributional JL Lemma:

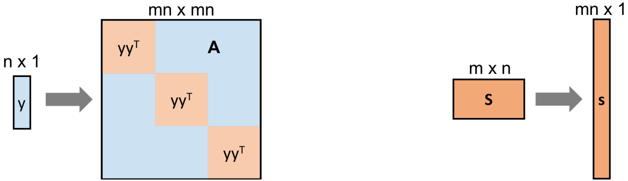
- Let $\mathbf{S} \in \mathbb{R}^{m \times n}$ have i.i.d. Gaussian entries. Observe that each entry of $\mathbf{S}y$ is distributed as $\mathcal{N}(0, \|y\|_2^2)$, and give a proof via concentration of independent Chi-Squared random variables (see 514 slides).
- Write $\|\mathbf{S}y\|_2^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{n} \mathbf{S}_{i,j}\mathbf{S}_{i,k}y_j y_k$ and prove concentration of this sum, even though the terms are not all independent of each other (only pairwise independent within one row).
- Apply the Hanson-Wright inequality – an exponential concentration inequality for random quadratic forms.
- This inequality comes up in a lot of places, including in the tight analysis of Hutchinson's trace estimator.

# Hanson Wright Inequality

**Theorem (Hanson-Wright Inequality)**

*Let $\mathbf{x} \in \mathbb{R}^n$ be a vector of i.i.d. random $\pm 1$ values. For any matrix $A \in \mathbb{R}^{n \times n}$,*

$$\Pr[\left|\mathbf{x}^T A \mathbf{x} - \operatorname{tr}(A)\right| \geq t] \leq 2 \exp\left(-c \cdot \min\left\{\frac{t^2}{\|A\|_F^2}, \frac{t}{\|A\|_2}\right\}\right).$$




Observe that $\mathbf{s}^T A \mathbf{s} = \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^n \mathbf{S}_{i,j} \mathbf{S}_{i,k} y_j y_k = \|\mathbf{S}y\|_2^2$ and that

$$\operatorname{tr}(A) = m \cdot \operatorname{tr}(yy^T) = m \cdot \|y\|_2^2.$$

# Distributional JL via Wright Inequality

Let $\mathbf{x} = \sqrt{m} \cdot \mathbf{s}$, so $\mathbf{x}$ has i.i.d. $\pm 1$ entries. Assume w.l.o.g. that $\|y\|_2 = 1$.

$$\begin{aligned}
\Pr\left[\left|\|\mathbf{S}y\|_2^2 - 1\right| \geq \epsilon\right] &= \Pr\left[\left|\mathbf{s}^T A \mathbf{s} - 1\right| \geq \epsilon\right] \\
&= \Pr\left[\left|\mathbf{x}^T A \mathbf{x} - m\right| \geq \epsilon m\right] \\
&= \Pr\left[\left|\mathbf{x}^T A \mathbf{x} - \mathrm{tr}(A)\right| \geq \epsilon m\right] \\
&\leq 2 \exp\left(-c \cdot \min\left\{\frac{(\epsilon m)^2}{\|A\|_F^2}, \frac{\epsilon m}{\|A\|_2}\right\}\right).
\end{aligned}$$

$\|A\|_F^2 = m \cdot \|yy^T\|_F^2 = m \cdot \|y\|_2^2 = m$

$\|A\|_2 = \|yy^T\|_2 = \|y\|_2 = 1$

$$\Pr\left[\left|\|\mathbf{S}y\|_2^2 - 1\right| \geq \epsilon\right] \leq 2 \exp\left(-c \cdot \min\left\{\frac{(\epsilon m)^2}{m}, \frac{\epsilon m}{1}\right\}\right) = 2\exp(-c\epsilon^2 m)$$

If we set $m = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)$, $\Pr\left[\left|\|\mathbf{S}y\|_2^2 - 1\right| \geq \epsilon\right] \leq \delta$, giving the distributional JL lemma.

# Questions?