COMPSCI 690RA: Randomized Algorithms and Probabilistic Data Analysis

Prof. Cameron Musco

University of Massachusetts Amherst. Spring 2022.

Lecture 12 (Final Lecture!)

Logistics

- The final exam is this Friday 5/6 at 10:30am in this room.
- I will hold extended office hours today from 2-4pm and tomorrow from 4-6pm.
- I will accept final project submissions up until Sunday 5/8 at 11:59pm.
- Please complete your SRTI for the class when you get a chance!

Summary

Last Week: Finish up Markov Chains Unit.

- Mixing time analysis via coupling.
- Example applications to shuffling and random walks on the hypercube.
- Markov Chain Monte Carlo (MCMC) methods.
- Example of reductions from counting to sampling (e.g., for counting independent sets).

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Today: The Probabilistic Method (not on the exam)

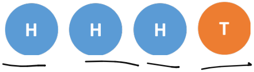
- From probabilistic proofs to algorithms via the method of conditional expectations.
- The Lovasz local lemma for events with 'bounded' correlation.
- Entropic proof of the algorithmic LLL.

First...a detour

The St. Petersburg Paradox

Consider the following game: you keep flipping a fair coin, until it hits tails. You win \$$2^{k+1}$, where k is the number of heads you see.

$2^4 = 16$



$2^1 = 2$

T

The St. Petersburg Paradox

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Let X be the amount of money you win. What is $\mathbb{E}[X]$?

$$\frac{1}{2}\cdot 2 + \frac{1}{4}\cdot 4 + \frac{1}{8}\cdot 8 = 1 + 1 + \dots 1 = \infty$$

The St. Petersburg Paradox

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1 + 1 + 1 + 1

Let X be the amount of money you win. What is $\mathbb{E}[X]$? $= \infty$

How much money would you pay to play this game? Why?

\$2

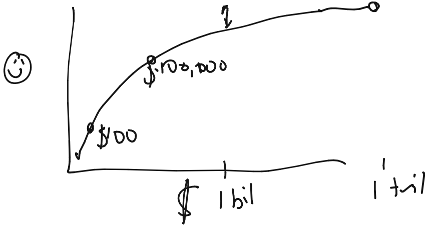
\$20

- don't care about low prob events
- finite amount of money

Expected Utility Theory

One Solution to the Paradox: The expected value of the game is not $\mathbb{E}[X]$, but $\mathbb{E}[U(X)]$ where U is some utility function.

$U(\cdot)$ determines how much actual value you derive from a given amount of money. We expect generally that U is concave – diminishing marginal utility.



Maximizing Expected Log Winnings

$:2$

What is $\mathbb{E}[U(X)] = \mathbb{E}[\log_2(X)]$ for our game?

$$\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \cdots$$

$$\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots\right) + \left(\frac{1}{4} + \frac{1}{8} + \cdots\right) + \left(\frac{1}{8} + \frac{1}{16} + \cdots\right)$$

$$1 + \frac{1}{2} + \frac{1}{4}$$

$$= 2$$

A More ‘Realistic’ Scenario

You are given \$25 and are allowed to play the following game repeatedly: You have a biased coin that hits heads 60% of the time. You can wager \$$w$ on if the coin hits heads or tails. If you are correct, you win \$$w$, and if you are incorrect, you lose \$$w$.

How should you determine the size of your bets?

you can play for 1 hour

\$25

1/2 your money.

$\mathbb{E}x_i$

$\mathbb{E}[\log(x_i)]$

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$$\mathbb{E}[\log(X_{i+1})|X_i] = .6 \cdot \overset{X_{i+1}}{\log(X_i + w)} + .4 \cdot \overset{X_{i+1}}{\log(X_i - w)}.$$

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Write $w = r \cdot X_i$. Then:

$$\mathbb{E}[\log(X_{i+1})|X_i] = .6 \cdot \log(X_i \cdot (1 + r)) + .4 \cdot \log(X_i \cdot (1 - r))$$

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$\lim_{n \to \infty} \frac{1}{n} \log\left(\frac{X_n}{X_{n+1}}\right)$

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10 vs 2 vs 1

$\log(5)$

$\log(2)$

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To maximize $.6 \log(1+r) + .4 \log(1-r)$, set its derivative to 0:

$$0 = \frac{.6}{1+r} - \frac{.4}{1-r}.$$

$$\frac{.4}{(1-r)} = \frac{.6}{1+r}$$

$$.4 + .4r = .6 - .6r$$

$$r = .6 - .4 = .2$$

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To maximize $.6 \log(1 + r) + .4 \log(1 - r)$, set its derivative to 0:

$$0 = \frac{.6}{1 + r} - \frac{.4}{1 - r}.$$

Optimal $r = 0.2$. I.e., you should bet 20% of your money each time.

The prior analysis is a special case of the Kelly criterion.

$$r = p - \frac{q}{b}.$$

Lots of interesting topics here, closely related to Markov chains and Martingales.

$.4 - .6 = -.2$

The Probabilistic Method

The Basic Idea: Suppose we want to prove that a combinatorial object satisfying a certain property exists. Then it suffices to exhibit a random process that produces such an object with probability > 0.

A common tool: For a random variable with $\mathbb{E}[X = \mu]$, $\Pr[X \geq \mu] > 0$ and $\Pr[X \leq \mu] > 0$.

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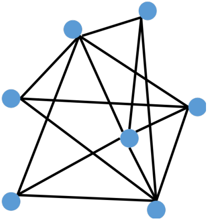

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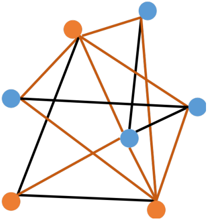

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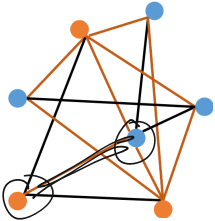

We have $\mathbb{E}[X] =$ $\frac{m}{2}$

$$\mathbb{E}\, X_{u,v} = \frac{1}{2}$$

$$\mathbb{E}\, X = \sum_{(u,v)} \mathbb{E} X_{u,v} = m \cdot \frac{1}{2} = \frac{m}{2}$$

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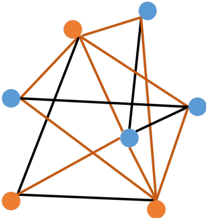

We have $\mathbb{E}[X] = \frac{m}{2}$

Therefore, $\Pr[X \geq m/2] > 0$. So every graph with m edges has a cut containing at least $m/2$ edges.

Example 2: 3-SAT

Prove that for any 3-SAT formula, there is some assignment of the variables such that at least 7/8 of the clauses are true.

$(x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_2 \vee x_4 \vee x_4) \ldots$

1 1 0

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Prove that for any 3-SAT formula, there is some assignment of the variables such that at least 7/8 of the clauses are true.

Consider a random assignment of the variables. And let X be the number of satisfied clauses.

$$\overset{\{0,1\}}{(x_1} \vee \bar{x}_2 \vee x_4) \wedge \overset{\{0,1\}}{(x_2} \vee \bar{x}_4 \vee x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge \ldots$$

$\underline{x_1} = 0$, $\bar{x}_2 = 1$, $x_4 = 0$

$\mathbb{E} X_i = \Pr(\text{clause } i \text{ is sat.})$
$= 7/8$

What is $\mathbb{E}[X]$?

$$\mathbb{E}[X] = \sum_{i=1}^{m} \mathbb{E} X_i$$

$$= \frac{7}{8} \cdot m$$

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What is $\mathbb{E}[X]$? $\underline{7/8\ m}$

So, $\Pr[X \geq 7/8m] > 0$. So there is an assignment satisfying at least 7/8 of the clauses in every 3-SAT formula.

From Existence to Efficient Algorithms

$\leq m$

Max-Cut Approximation: A randomly sampled partition cuts $m/2$ edges in expectation. But how many partitions do we need to sample before finding a cut of size at least $m/2$ with good probability?

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Let p be the probability of finding a cut of size $\geq m/2$. Then:

$$\mathbb{E}[X] = \frac{m}{2} \leq (1-p) \cdot \left(\frac{m}{2} - 1\right) + p \cdot m$$

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$$\Omega\left(\frac{1}{m}\right) \qquad \implies \frac{1}{\frac{m}{2}+1} \leq p.$$

$$\cancel{\frac{m}{2}} \leq \cancel{\frac{m}{2}} - 1 - \frac{pm}{2} + p + pm$$

$$1 \leq p \cdot \left(\frac{m}{2} + 1\right)$$

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$$\implies \underbrace{\frac{1}{\frac{m}{2}+1}}_{} \leq p. \qquad \Omega\left(\frac{1}{m}\right)$$

How many attempts do we need to take to find a large cut with probability at least $1 - \delta$?

$$O(\underbrace{m \log(1/\delta)}_{})$$

$$\left(1 - \frac{1}{m}\right)^{m \log(1/\delta)} \approx \underline{\delta}$$

From Existence to Efficient Algorithms

Max-Cut Approximation: A randomly sampled partition cuts $m/2$ edges in expectation. But how many partitions do we need to sample before finding a cut of size at least $m/2$ with good probability?

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$$\implies \frac{1}{\frac{m}{2}+1} \leq p.$$

How many attempts do we need to take to find a large cut with probability at least $1 - \delta$? $O(m \cdot \log(1/\delta))$

Method of Conditional Expectations

We can also derandomize this algorithm in a very simple way.

Let $x_1, x_2, \ldots \in \{0, 1\}$ indicate if the vertices are included on one side of the random partition.

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Consider determining these random variables sequentially.

$$\frac{m}{2} = \mathbb{E}[X] = \frac{1}{2}\mathbb{E}[X|x_1 = 1] + \frac{1}{2}\mathbb{E}[X|x_1 = 0].$$

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Set $x_1 = \overset{\{0,1\}}{v_1}$ such that $\mathbb{E}[X|x_1 = v_1] \geq \frac{m}{2}$ Then we have:

$$\frac{m}{2} \leq \mathbb{E}[X|x_1 = v_1] = \frac{1}{2}\mathbb{E}[X|x_1 = v_1, x_2 = 1] + \frac{1}{2}\mathbb{E}[X|x_1 = v_1, x_2 = 0]$$

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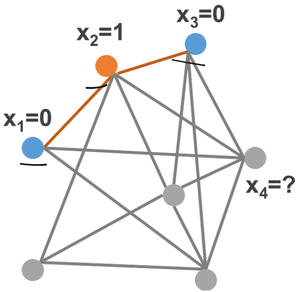
Set $x_2 = v_2$ such that $\mathbb{E}[X|x_1 = v_1, x_2 = v_2] \geq \frac{m}{2}$. And so on...

Conditional Expectations for Cuts

How can we pick v_i such that $\mathbb{E}[X|x_1 = v_1, \ldots, x_{i\cancel{-1}} = v_{i\cancel{-1}}] \geq \frac{m}{2}$?



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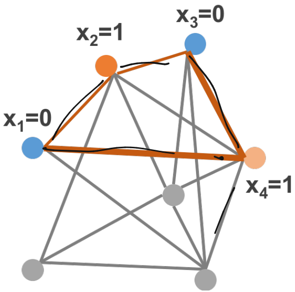

$\mathbb{E}[X|x_1 = 0, \ldots, x_4 = 1] \overset{+2}{=} \underbrace{\frac{1}{2} \cdot 10}\underbrace{+ 2} = 9$

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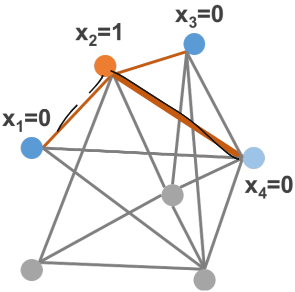

$\mathbb{E}[X|x_1 = 0, \ldots, x_4 = 0] = \overset{2+}{\frac{1}{2}} \cdot 10 + 1 = 8$

Conditional Expectations for Cuts

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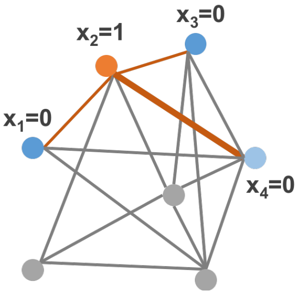

Natural greedy approach: add vertex i to the side of the cut to which it has fewest edges.

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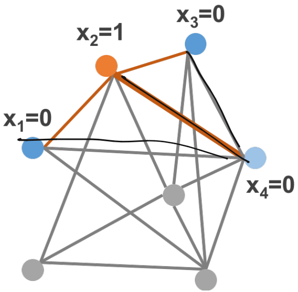

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Yields a 1/2 approximation algorithm for max-cut.

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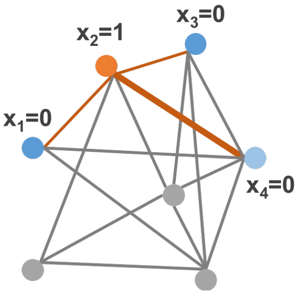
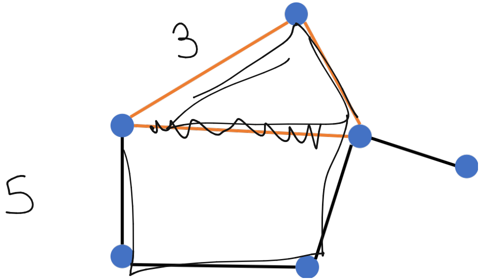


Natural greedy approach: add vertex i to the side of the cut to which it has fewest edges.

Yields a 1/2 approximation algorithm for max-cut. 16/17 is the best possible assuming $P \neq NP$, and .878 is the best known (Goemans, Williamson) and optimal assuming the unique games conjecture.

Large Girth Graphs

The girth of a graph is the length of its shortest cycle.



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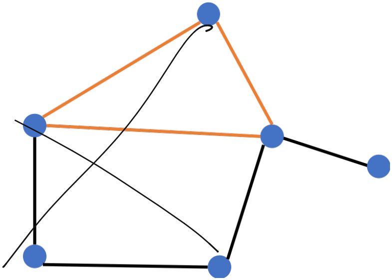
3

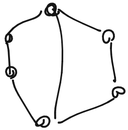


Natural Question: How large can the girth be for a graph with *m* edges?

$\binom{n}{2}$

n edges

$n+1$ edges



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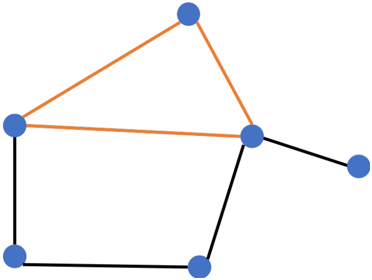
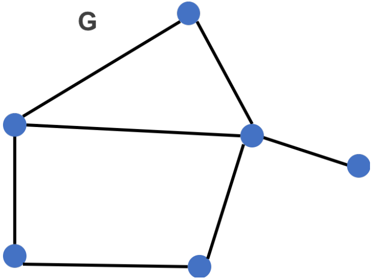


Natural Question: How large can the girth be for a graph with m edges?

Erdös Girth Conjecture: For any $k \geq 1$, there exists a graph with $m = \Omega(n^{1+1/k})$ edges and girth $2k+1$.

Relevance to Spanners

A spanner is a subgraph that approximately preserves shortest path distances. We say G' is a spanner for G with stretch t if for all u, v $d_{G'}(u, v) \leq t \cdot d_G(u, v)$.



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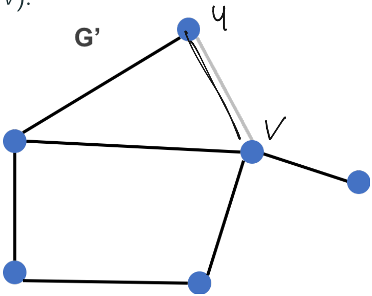

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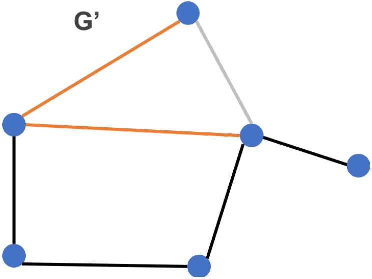
$t \geq 2$



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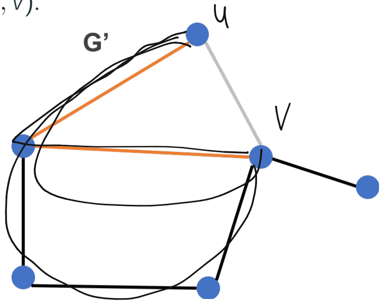

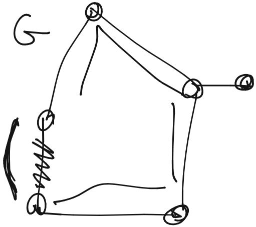

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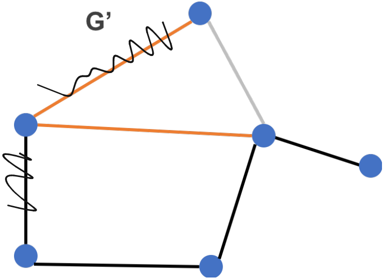

Even when G' excludes a single edge, $t \geq girth(G) - 1$.

Erdös Girth Conjecture $\implies$ there are no generic spanner constructions with $o(n^{1+1/k})$ edges and stretch $\leq 2k - 1$.

Large Girth Graphs via Probabilistic Method

Theorem

For any fixed $k \geq 3$, there exists a graph with n nodes, $\Omega(n^{1+1/k})$ edges, and girth $k+1$.

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$p(n-1) \approx n^{1/k}$

Sample and Modify Approach: Let G be an Erdös-Renyi random graph, where each edge is included independently with probability $p = n^{1/k-1}$. Remove one edge from every cycle in G with length $\leq k$, to get a graph with girth $k+1$.

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$$\mathbb{E}[X] = p \cdot \binom{n}{2}$$

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$$\mathbb{E}[X] = \frac{n(n-1)}{2} \cdot p = \frac{1}{2} \cdot \overbrace{\left(1 - \frac{1}{n}\right)}^{\sim 1} \cdot n^{1+1/k}.$$

$$\frac{n(n-1)}{2} \cdot n^{1/k-1} = \frac{n-1}{2} \cdot n^{1/k}$$

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1 2 3 4

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$\leq n$

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$$\frac{n \cdot (n-1) \ldots (n-i+1)}{i \cdot (i-1) \ldots 1}$$

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$$(n \cdot p)^i = (n \cdot n^{1/k-1})^i = (n^{1/k})^i = n^{i/k}$$

So far: An Erdös-Renyi random graph with $p = n^{1/k-1}$ has expected number of edges (X) and cycles of length $\leq k-1$ (Y) bounded by:

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When k is fixed and n is sufficiently large, $k \cdot n \ll n^{1+1/k}$. Thus,

$$\mathbb{E}[X - Y] = \Omega(\mathbb{E}[X]) = \Omega(n^{1+1/k}),$$

proving the theorem.

Lovasz Local Lemma

Probabilities of Correlated Events

Suppose we want to sample a random object that avoids n 'bad events' $E_1, \ldots, E_n$.

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As n gets large, the union bound gets very weak – each event has to occur with probability $< 1/n$ on average.

Bounded Correlation

Consider events $E_1, \ldots, E_n$ where E_i is independent of any E_j for $j \notin \Gamma(i)$ (the neighborhood of i in the dependency graph)

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$$\underbrace{(x_1 \vee \bar{x}_2 \vee x_3)}_{E_1} \wedge \underbrace{(x_2 \vee \bar{x}_4 \vee x_3)}_{E_2} \wedge \underbrace{(x_4 \vee x_5 \vee x_6)}_{E_3} \wedge \underbrace{(\neg x_4 \vee x_6 \vee x_7)}_{E_4} \ldots$$

$|\Gamma(1)| = 1$

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Theorem (Lovasz Local Lemma) LLL

Suppose for a set of events $E_1, E_2, \ldots, E_n$, $\Pr[E_i] \leq p$ for all i, and that each E_i is dependent on at most d other events E_j (i.e., $|\Gamma(i)| \leq d$, then if $4dp \leq 1$:

$(n-1)p < 1$

dp

$$\Pr\left[\neg \bigcup_{i=1}^{n} E_i\right] > (1-2p)^n > 0.$$

In the worse case, $d = n - 1$ and this is similar to the union bound. But it can be much stronger.

LLL Application: k-SAT

Theorem

If no variable in a k-SAT formula appears in more than $\frac{2^k}{4k}$ clauses, then the formula is satisfiable.

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LLL Application: k-SAT

$\overset{0}{x_1} \vee \overset{1}{\bar{x}_2} \vee \overset{0}{x_3}$

$\frac{64}{24} \approx 2$

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LLL Application: k-SAT

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correlated with E_i

events

Let E_i be the event that clause i is unsatisfied by a random assignment. $\Pr[E_i] \leq \frac{1}{2^k} = p$.

$|\Gamma(i)| \leq \binom{n}{k} \cdot 2^k$

$\leq k \cdot \frac{2^k}{4k} = \frac{2^k}{4}$

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So $4dp = 4 \cdot \frac{1}{2^k} \cdot \frac{2^k}{4} \le 1$, and thus $\Pr\left[\neg \bigcup_{i=1}^n E_i\right] > 0$. I.e., a random assignment satisfies the formula with non-zero probability.

Algorithmic LLL

Important Question: Given an Lovasz Local Lemma based proof of the existence, can we convert it into an efficient algorithm?

Moser and Tardos [2010] prove that a very natural algorithm can be used to do this.

Let $E_1, \ldots, E_n$ be events determined by a set of independent random variables $V = \{v_1, \ldots, v_m\}$. Let $v(E_i)$ be the set of variables that E_i depends on.

$(x_1 \vee x_2 \vee \bar{x}_3)$

Resampling Algorithm:

1. Assign $v_1, \ldots, v_m$ random values.
2. While there is some E_i that occurs, reassign random values to all varables in $v(E_i)$.
3. Halt when an assignment is found such that no E_i occurs.

Algorithmic LLL

Theorem (Algorithmic Lovasz Local Lemma)

Consider a set of events $E_1, E_2, \ldots, E_n$ determined by a finite set of random variables V. If for all i, $\Pr[E_i] \leq p$ and $|\Gamma(i)| \leq d$, and if $ep(d+1) \leq 1$, then Resampling *finds an assignment of the variables in V such that no event E_i occurs. Further, the algorithm makes $O(\frac{n}{d})$ iterations in expectation.*

$4pd \leq 1$

$n \cdot p = \Theta(n/d)$

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$$\Pr[E_i] \leq \frac{1}{2^k} = p \quad \text{and} \quad |\Gamma(i) \leq k \cdot \frac{2^k}{5k} = \frac{2^k}{5} = d.$$

Have $\underline{ep(d+1)} \leq \frac{e}{5} + \frac{e}{2^k} \leq 1$ as long as $k \geq 3$, so the theorem applies, giving a polynomial time algorithm for this variant of k-SAT.

Entropic Proof of Algorithmic LLL

Moser's 'entropic proof' of the algorithmic LLL uses a particularly cool technique.

Focus on the case of k-SAT where $|\Gamma(i)| < d = \frac{2^k}{8} = 2^{k-3}$.

- In each iteration of rerandomization, the algorithm uses k random bits. So for T iterations it uses Tk random bits.
- We will show that if we run the algorithm for too long, then we obtain a compression of these bits into a string of $< Tk$ bits, which shouldn't be possible (since they are random bits and incompressible).

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$\underline{2^t}$ $\quad 1+2+4+\ldots 2^{t-c}$

$\leq 2^{(t-c)+1}$

Incompressibility Fact: For any function f mapping inputs in $\{0,1\}^t$ to distinct, possibly variable length binary output strings, if s is a uniform random t-bit binary string, then for any integer c, $\Pr[length[f(s)] \leq \underline{t-c}] \leq \frac{1}{2^{c-1}}$.

$$\frac{2^{(t-c)+1}}{2^t} = \frac{1}{2^{c-1}}$$

Compressing Bits While Solving k-SAT

- Initialize random assignments for the m variables using m bits.
- Iterate through the clauses, recording '1' for each that is satisfied, and recording '0' when you reach an unsatisfied clause i.
- Run LocalCorrect(i). Then move on to the next clause.
- After completion of all clauses, record the final state of the m variables using m bits.

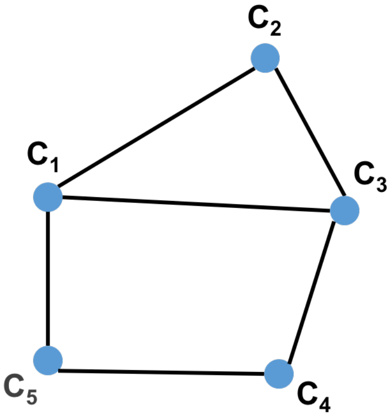
LocalCorrect(i):

- Resample random values for the variables in clause i, using k bits (but don't record them!).

$\log(2^k/8) = k-3$

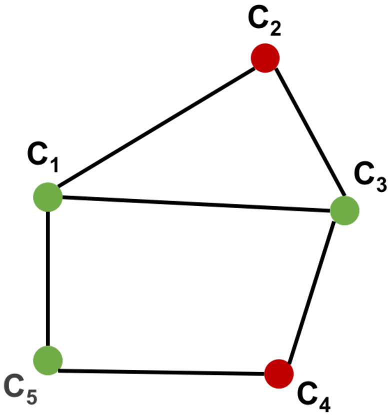
- While some clause $j \in \Gamma(i) \cup \{i\}$ is unsatisfied, pick the first such j, and record '0' along with j using $k-3$ bits. Then run LocalCorrect(j).
- Record '1' upon termination.

Compression Illustration



Record:
Bits Used:

Compression Illustration



Record:
Bits Used: 10110001

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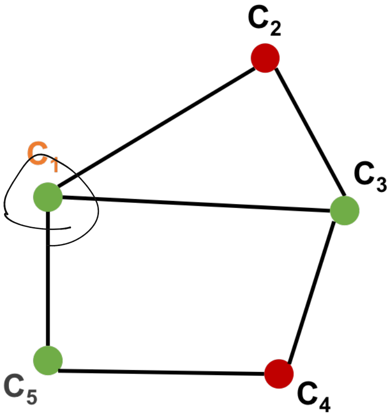
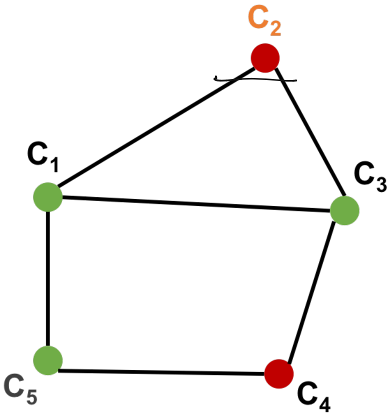


Record: 1

Bits Used: 10110001

Compression Illustration



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Bits Used: 10110001

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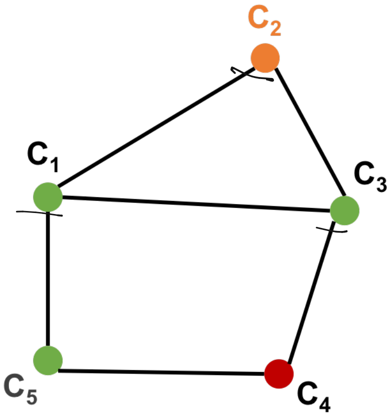

Record: 10

Bits Used: 1011000110111

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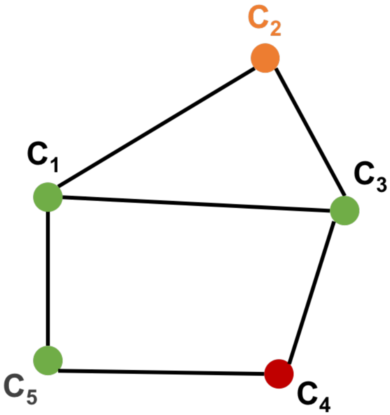
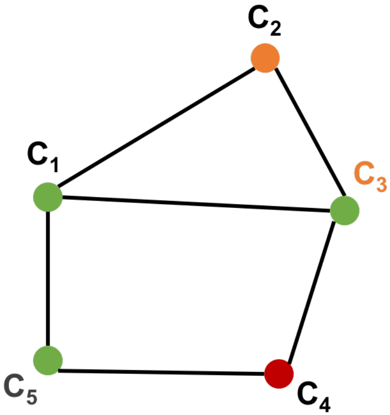


Record: 101

Bits Used: 1011000110111

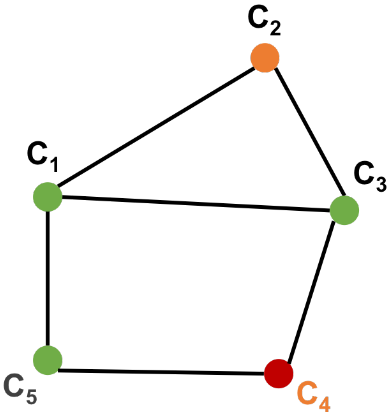
Compression Illustration



Record: 1011

Bits Used: 1011000110111

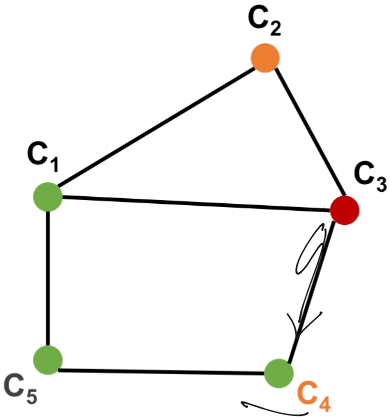
Compression Illustration



Record: 10110

Bits Used: 1011000110111

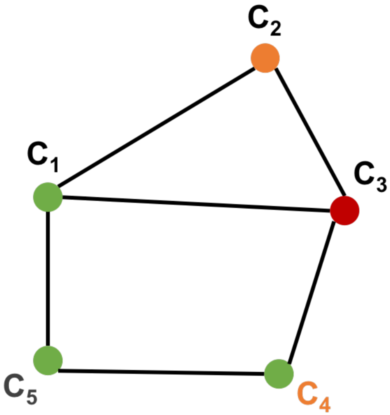
Compression Illustration



Record: 10110

Bits Used: 101100011011100100

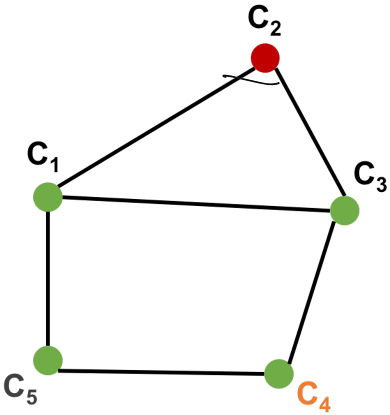
Compression Illustration



Record: 10110000

Bits Used: 1011000110111100100

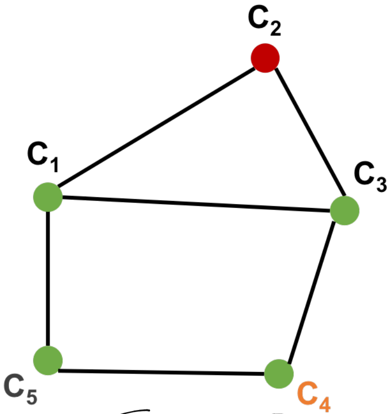
Compression Illustration



Record: 10110000

Bits Used: 1011000110111001001010101

Compression Illustration



Record: 10110000001 1100100

Bits Used: 10110001101110010010101

Compressing Bits While Solving k-SAT

Claim 1: If the algorithm runs for T iterations, it uses $m + Tk$ random bits.

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2^{k-5}

2^{k-10}

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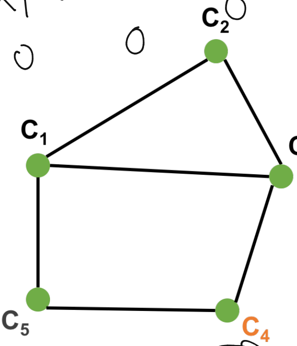
So we have compressed $B = m + Tk$ bits to $B' = m + n + T(k-1)$ bits. Setting, $T = n + c$ we have $B = m + Tk$ and $B' = m + Tk - c$. Thus, by our incompressibility claim, the algorithm must terminate in $n + c$ steps with probability $1 - 1/2^{c-1}$.

Recovery Illustration

$x_1 \vee x_2 \vee x_3$

0 0 0



Record: 10110000001...10111011

Bits Used: 10110001,10111,00100,10101,10111

Thanks for a great semester!