COMPSCI 614: Randomized Algorithms with Applications to Data Science

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Lecture 20

Summary

Last Time: Markov Chain Fundamentals

- The gambler's ruin problem.
- Aperiodicity and stationary distribution of a Markov chain.
- The fundamental theorem of Markov chains.
- Example of a uniform stationary distribution for a symmetric Markov chain (shuffling).

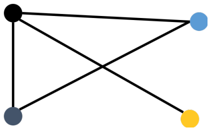
Today: Mixing Time Analysis

- How quickly does a Markov chain actually converge to its stationary distribution?
- Mixing time and its analysis via coupling.

Stationary Distribution Example 2

Random Walk on an Undirected Graph: Consider a random walk on an undirected graph. If it is at node i at step t, then it moves to any of i's neighbors at step $t+1$ with probability $\frac{1}{d_i}$.

- What is the state space of this chain?
- What is the transition probability $P_{i,j}$?
- Is this chain aperiodic?
- If the graph is not bipartite, then there is at least one odd cycle, making the chain aperiodic.



Stationary Distribution Example 2

Random Walk on an Undirected Graph: Consider a random walk on an undirected graph. If it is at node i at step t, then it moves to any of i's neighbors at step $t+1$ with probability $\frac{1}{d_i}$.

Claim: When the graph is not bipartite, the unique stationary distribution of this Markov chain is given by $\pi(i) = \frac{d_i}{2|E|}$.

$$\pi P_{:,i} = \sum_j \pi(j) P_{j,i} = \sum_j \frac{d_j}{2|E|} \cdot \frac{1}{d_j} = \sum_j \frac{1}{2|E|} = \frac{d_i}{2|E|} = \pi(i).$$

I.e., the probability of being at a given node i is dependent only on the node's degree, not on the structure of the graph in any other way.

What is the stationary distribution over the edges?

Mixing Times

Total Variation Distance

Definition (Total Variation (TV) Distance)

For two distributions $p, q \in [0,1]^m$ over state space $[m]$, the total variation distance is given by:

$$\|p - q\|_{TV} = \frac{1}{2} \sum_{i \in [m]} |p(i) - q(i)| = \max_{A \subseteq [m]} |p(A) - q(A)|.$$

Kontorovich-Rubinstein duality: Let P, Q be possibly correlated random variables with marginal distributions p, q. Then

$$\|p - q\|_{TV} \leq \Pr[\mathsf{P} \neq \mathsf{Q}].$$

Mixing Time

Definition (Mixing Time)

Consider a Markov chain $X_0, X_1, \ldots$ with unique stationary distribution π. Let $q_{i,t}$ be the distribution over states at time t assuming $X_0 = i$. The mixing time is defined as:

$$\tau(\epsilon) = \min\left\{t : \max_{i \in [m]} \|q_{i,t} - \pi\|_{TV} \leq \epsilon\right\}.$$

I.e., what is the maximum time it takes the Markov chain to converge to within ϵ in TV distance of the stationary distribution?

Note: If $\|q_{i,t} - \pi\|_{TV} \leq \epsilon$ then for any $t' \geq t$, $\|q_{i,t'} - \pi\|_{TV} \leq \epsilon$.

Mixing Time Convergence

Typically, it suffices to focus on the mixing time for $\epsilon = 1/2$. We have:

Claim: If $X_0, X_1, \ldots$ is finite, irreducible, and aperiodic, then $\tau(\epsilon) \leq \tau(1/2) \cdot c \log(1/\epsilon)$ for large enough constant c.

Coupling Motivation

Claim: $\max_{i\in[m]} \|q_{i,t} - \pi\|_{TV} \leq \max_{i,j\in[m]} \|q_{i,t} - q_{j,t}\|_{TV}$.

$$
\begin{aligned}
\|q_{i,t} - \pi\|_{TV} &= \|q_{i,t} - \pi P^t\|_{TV} \\
&= \|q_{i,t} - \sum_j \pi(j) e_j P^t\|_{TV} \\
&= \|q_{i,t} - \sum_j \pi(j) q_{j,t}\|_{TV} \\
&\leq \sum_j \|\pi(j) q_{i,t} - \pi(j) q_{j,t}\|_{TV} \\
&\leq \sum_j \pi(j) \cdot \|q_{i,t} - q_{j,t}\|_{TV} \\
&\leq \max_{j\in[m]} \|q_{i,t} - q_{j,t}\|_{TV}.
\end{aligned}
$$

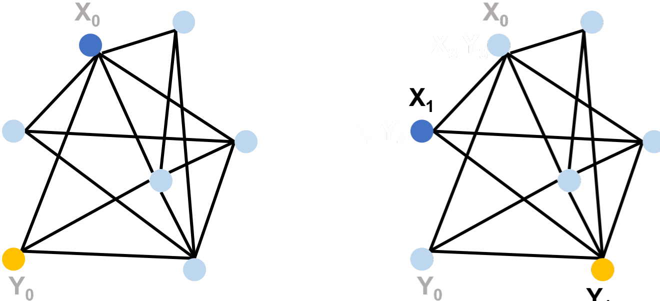
Coupling: A common technique for bounding the mixing time by showing that $\max_{i,j\in[m]} \|q_{i,t} - q_{j,t}\|_{TV}$ is small.

Formal Coupling Definition

Definition (Coupling)

For a finite Markov chain $X_0, X_1, \ldots$ with transition matrix $P \in \mathbb{R}^{m \times m}$, a coupling is a joint process $(X_0, Y_0), (X_1, Y_1), \ldots$ such that:

1. $X_0 = i$ and $Y_0 = j$ for some $i, j \in [m]$.
2. $\Pr[X_t = j | X_{t-1} = i] = \Pr[Y_t = j | Y_{t-1} = i] = P_{i,j}$
3. If $X_t = Y_t$, then $X_{t+1} = Y_{t+1}$.



Coupling Theorem Proof

Theorem (Mixing Time Bound via Coupling)

For a finite, irreducible, and aperiodic Markov chain $X_0, X_1, \ldots$ *and any valid coupling* $(X_0, Y_0), (X_1, Y_1), \ldots$ *letting* $T_{i,j} = \min\{t : X_t = Y_t | X_0 = i, Y_0 = j\}$,

$$\max_{i \in [m]} \|q_{i,t} - \pi\|_{TV} \leq \max_{i,j \in [m]} \|q_{i,t} - q_{j,t}\|_{TV} \leq \max_{i,j \in [m]} \Pr[T_{i,j} > t].$$

Follows from Kontorovich-Rubinstein duality.

For X_t, Y_t distributed by evolving the chain for t steps starting from state i or j respectively, we have:

$$\max_{i,j \in [m]} \|q_{i,t} - q_{j,t}\|_{TV} \leq \max_{i,j \in [m]} \Pr[X_t \neq Y_t] = \max_{i,j \in [m]} \Pr[T_{i,j} > t]$$

Coupling Example: Mixing Time of Shuffling

How many times do we need to swap a random card to the top of the deck so that the distribution of orderings on our cards is ϵ-close in TV distance to the uniform distribution over all permutations?

Coupling:

- Let $X_0, X_1, \ldots$ be the Markov chain where a random card is moved to the top in each step.
- Let Y_0, Y_1 be a correlated Markov chain. When card S is swapped to the top in the X chain, swap S to the top in the Y chain as well.
- Can check that this is a valid coupling since X_t, Y_t have the correct marginal distributions, and since $X_t = Y_t \implies X_{t+1} = Y_{t+1}$
- Observe that $X_t = Y_t$ as soon as all c unique cards have been swapped at least once. How many swaps does this take?

X_0 Y_0

Coupling Example: Mixing Time of Shuffling

$$\max_{i\in[m]} \|q_{i,t} - \pi\|_{TV} \le \max_{i,j\in[m]} \Pr[\mathrm{T}_{i,j} > t]$$
$$\le \Pr[< c \text{ unique cards are swapped in } t \text{ swaps}]$$

By coupon collector analysis for $t \ge c\ln(c/\epsilon)$, this probability is bounded by ϵ. In particular, by the fact that $\left(1-\frac{1}{c}\right)^{c\ln c/\epsilon} \le \frac{\epsilon}{c}$ plus a union bound over c cards.

Thus, for $t \ge c\ln(c/\epsilon)$,
$\max_{i\in[m]} \|q_{i,t} - \pi\|_{TV} \le \max_{i,j\in[m]} \|q_{i,t} - q_{j,t}\|_{TV} \le \epsilon$.

I.e., $\tau(\epsilon) \le c\ln(c/\epsilon)$.