COMPSCI 614: Randomized Algorithms with Applications to Data Science

Prof. Cameron Musco

University of Massachusetts Amherst. Spring 2024.

Lecture 14 ?

Logistics

- I'll return midterms at the end of class.
- Overall the class did well – mean was a 25.5 out of 34 ($\approx$ 75%).
- Generally speaking people felt the test was a bit rushed.
- If you are not happy with your performance, message me and we can chat about it. I'm also happy to review solutions in office hours.
- I plan to release Problem Set 4 by end of this week.
- 2 page progress report on Final Project due 4/16.

Summary

Randomized Linear Algebra Before Break: , importance sampling

- Approximate matrix multiplication via norm-based sampling. Analysis via outer-product view of matrix multiplication.
- Application to fast randomized low-rank approximation.
- Hutchinson's method for trace estimation. Analysis via linearity of variance for pairwise-independent random variables.
- Random linear sketching for ℓ_0 sampling and ℓ_2 heavy-hitters (Count Sketch).

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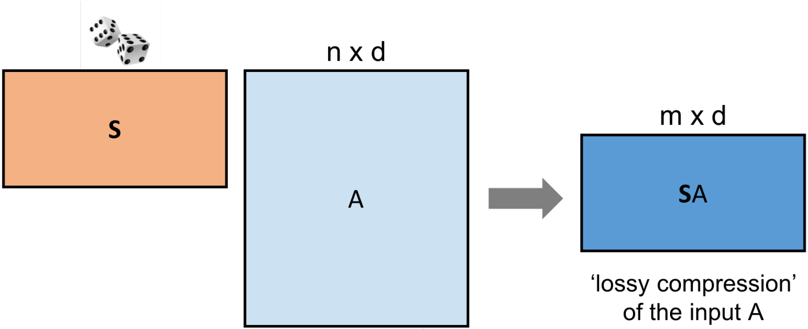
Today:

- Linear sketching for dimensionality reduction and the Johnson-Lindenstrauss lemma.
- Subspace embedding and ϵ-net arguments.

└ learning theory
random matrix . . .

Linear Sketching

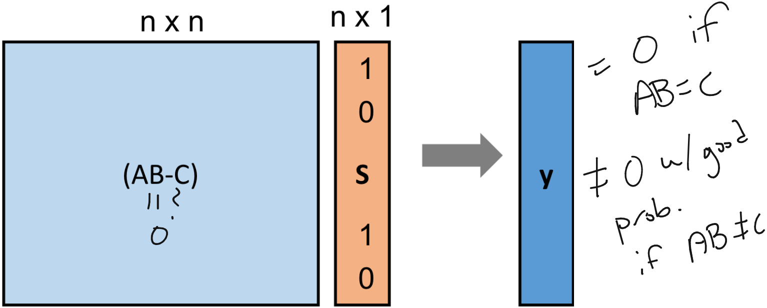
Given a large matrix $A \in \mathbb{R}^{n \times d}$, we pick a random linear transformation $\mathbf{S} \in \mathbb{R}^{m \times n}$ and compute $\mathbf{S}A$ (alternatively, pick $\mathbf{S} \in \mathbb{R}^{d \times m}$ and compute $A\mathbf{S}$). Using $\mathbf{S}A$ we can approximate many computations involving A.



linear ⇒ works in streaming + distributed
⇒ very fast

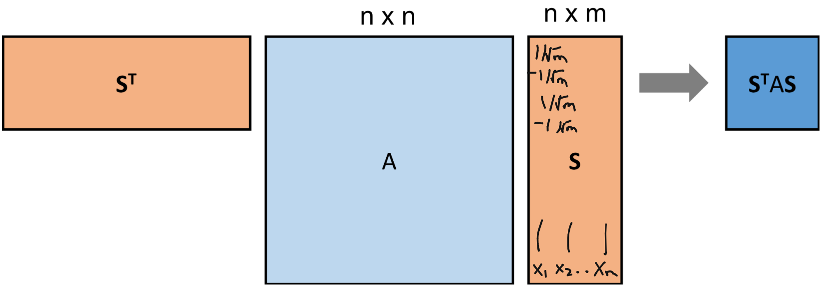
, Homework 1

Freivald's Algorithm:



Linear Sketching Examples

Hutchinson’s Trace Estimator:



$$\mathrm{tr}(A) \approx \frac{1}{m}\sum_{i=1}^{m} x_i^T A x_i = \mathrm{tr}(S^T A S)$$

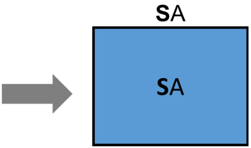
Graph Connectivity via ℓ_0 sampling:

l_0 sampling matrix **S**

1	**-1**	**0**	**0**	**1**	**-1**	**0**	**1**
-1	**0**	**1**	**1**	**0**	**0**	**-1**	**0**
1	**1**	**-1**	**0**	**-1**	**-1**	**0**	**1**
0	**-1**	**-1**	**-1**	**1**	**1**	**1**	**0**

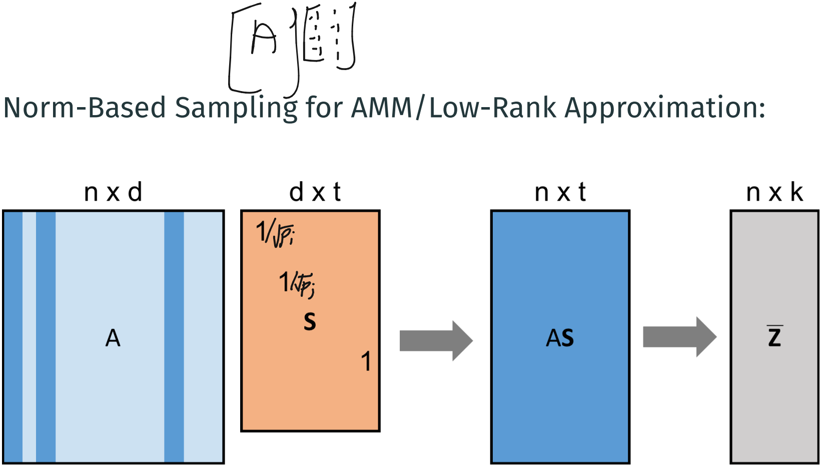
v_1	v_2	v_3	v_4
1	-1	0	0
0	1	0	-1
0	0	1	-1
-1	0	1	0
1	0	-1	0
0	1	-1	0
1	0	0	-1
0	0	1	-1

vertex-edge incidence matrix A



Linear Sketching Examples

Norm-Based Sampling for AMM/Low-Rank Approximation:



S depends on A

"non-oblivious" sketch

Subspace Embedding

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It is helpful to define general guarantees for sketches, that are useful in many problems.

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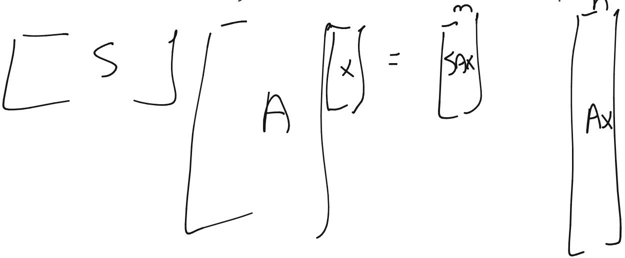
Definition (Subspace Embedding)

$S \in \mathbb{R}^{m \times d}$ is an ϵ-subspace embedding for $A \in \mathbb{R}^{n \times d}$ if, for all $x \in \mathbb{R}^d$,

$$(1-\epsilon)\|Ax\|_2 \leq \|SAx\|_2 \leq (1+\epsilon)\|Ax\|_2.$$

I.e., S preserves the norm of any vector Ax in the column span of A.



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$$\frac{1}{1+\varepsilon}\|SAx\| \leq \|Ax\| \approx (1-\varepsilon)\|SAx\|_2 \leq \|Ax\|$$

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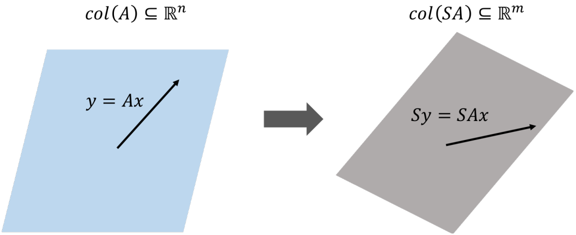

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$$(1-\epsilon)\|Ax\|_2 \leq \|SAx\|_2 \leq (1+\epsilon)\|Ax\|_2.$$

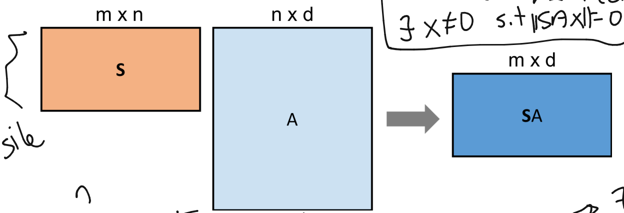
I.e., S preserves the norm of any vector Ax in the column span of A. Tons of applications. E.g.,

- Fast linear regression (next class) and preconditioning.
- Approximation of A's singular values.
- Approximate matrix multiplication and near optimal low-rank approximation.
- Compressed sensing/sparse recovery (related to ℓ_0 sampling).

Subspace Embedding Intuition

Think-Pair-Share 1: Assume that $n > d$ and that $\textbf{rank}(A) = d$. If $S \in \mathbb{R}^{m \times n}$ an is an ϵ-subspace embedding for A with $\epsilon < 1$, how large must m be? **Hint:** Think about $\textbf{rank}(SA)$ and/or the nullspace of SA.

We need $m \geq d$ because otherwise $\text{rank}(SA) \leq m < d$
$\exists\, x \neq 0$ s.t. $\|SAx\| = 0$ but $\|Ax\| > 0$
so don't have subspace embedding



n is possible

1 is impossible

$\exists x$ s.t.
$x \neq 0$
$\|SAx\| = 0$
$\|Ax\| > 0$
(A is full rank)

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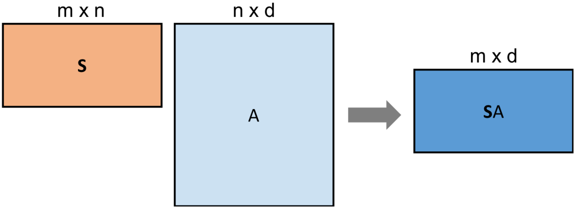
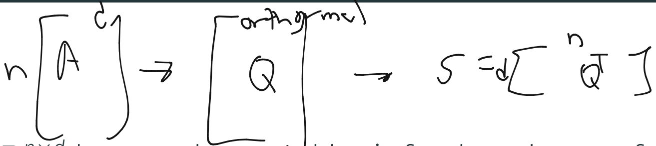


Think-Pair-Share 2: Describe how to deterministically compute a subspace embedding S with $m = d$ and $\epsilon = 0$ in $O(nd^2)$ time.

(we'll show $m \simeq d$ w/ randomized embeddings much faster

Let $Q \in \mathbb{R}^{n \times d}$ be an orthonormal basis for the columns of A. Then any vector Ax in A's column span can be written as Qy for some $y \in \mathbb{R}^d$.



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Let $S = Q^T$. $S \in \mathbb{R}^{d \times n}$ (i.e., $m = d$) and further, for any $x \in \mathbb{R}^d$

$$\|SAx\|_2^2 = \|Q^T Qy\|_2^2 \quad .$$

Optimal Subspace Embedding

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Optimal Subspace Embedding

$S = Q^T = [1]$ $A = [10]$ $Ax = 10x$

$SAx = 1 \cdot 10 \cdot x = Ax$

$S = A^+ = [10]$ $Q^T = (A^TA)^{-1/2} A^T$

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$$\|SAx\|_2^2 = \|Q^TQy\|_2^2 = \|y\|_2^2 = \|Ax\|_2^2.$$

$\|Ax\|_2^2 = \|Qy\|_2^2$

y^TQ^TQy

y^Ty

$= \|y\|_2^2$

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How would you compute Q?

↳ qr decomp.

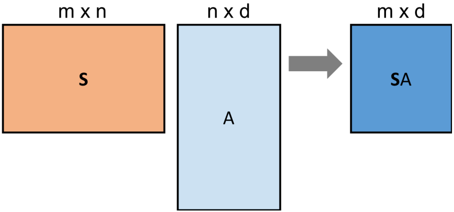
↳ gram schmidt (orth)

↳ SVD, inverse of $A^T A$

Randomized Subspace Embedding

Theorem (Oblivious Subspace Embedding)

Let $\mathbf{S} \in \mathbb{R}^{m \times d}$ *be a random matrix with i.i.d.* $\pm 1/\sqrt{m}$ *entries. Then if* $m = O\left(\frac{d + \log(1/\delta)}{\epsilon^2}\right)$, *for any* $A \in \mathbb{R}^{n \times d}$, *with probability* $\geq 1 - \delta$, $\mathbf{S}$ *is an* ϵ*-subspace embedding of* A.



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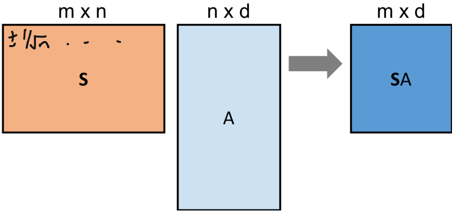
SA takes $O(ndm) \geq O(nd^2)$

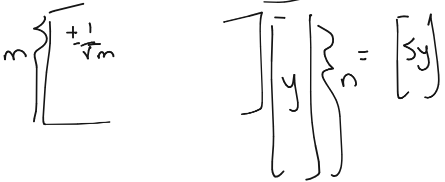


- $\mathbf{S}$ can be computed without any knowledge of A.
- Still achieves near optimal compression.
- Constructions where $\mathbf{S}$ is sparse or structured, allow efficient computation of $\mathbf{S}A$ (fast JL-transform, input-sparsity time algorithms via Count Sketch)

Oblivious Subspace Embedding Proof

1. **Distributional Johnson-Lindenstrauss**: For $\mathbf{S} \in \mathbb{R}^{m \times d}$ with i.i.d. $\pm 1/\sqrt{m}$ entries, for any fixed $y \in \mathbb{R}^n$, with probability $1 - \delta$ for very small δ, $(1-\epsilon)\|y\|_2 \leq \|\mathbf{S}y\|_2 \leq (1+\epsilon)\|y\|_2$.

for $m = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)$



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2. Via a union bound, have that for any fixed set of vectors $\mathcal{N} \subset \mathbb{R}^n$, with probability $1 - |\mathcal{N}| \cdot \delta$, $\|\mathbf{S}y\|_2 \approx_\epsilon \|y\|_2$ for all $y \in \mathcal{N}$.

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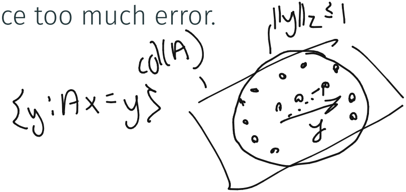

$y = A_{:,1} - A_{:,2} \approx 0$

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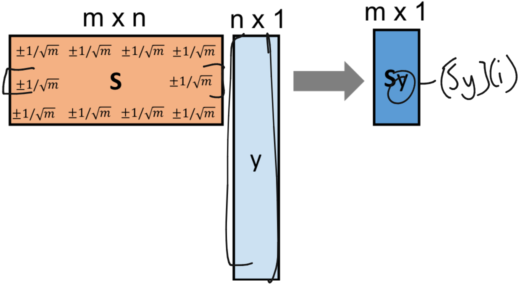
Remark: ϵ-nets are a key proof technique in theoretical computer science, learning theory (generalization bounds), random matrix theory, and beyond. They are a key take-away from this lecture.

Step 1: Distributional JL Lemma

Theorem (Distributional JL)

Let $\mathbf{S} \in \mathbb{R}^{m \times d}$ be a random matrix with i.i.d. $\pm 1/\sqrt{m}$ entries. Then if $m = O(\log(1/\delta)/\epsilon^2)$, for any fixed $y \in \mathbb{R}^n$, with probability $\geq 1 - \delta$, $(1-\epsilon)\|y\|_2 \leq \|\mathbf{S}y\|_2 \leq (1+\epsilon)\|y\|_2$.

I.e., via a random matrix, we can compress any vector from n to $\approx \log(1/\delta)/\epsilon^2$ dimensions, and approximately preserve its norm. A bit surprising maybe that m does not depend on n at all.



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$$\mathbb{E}[\|\mathsf{S}y\|_2^2] = \sum_{i=1}^{m} \mathbb{E}[\langle \mathsf{S}_{i,:}, y \rangle^2]$$

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$\mathbb{E} \sum S_{ij} y_i \stackrel{?}{=} 0$

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$$= \sum_{i=1}^{m}\sum_{j=1}^{n} \mathrm{Var}(\mathbf{S}_{ij} \cdot y_j)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{1}{m} \cdot y_j^2$$

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$$= \sum_{i=1}^{m}\sum_{j=1}^{n} \mathrm{Var}(\mathbf{S}_{ij} \cdot y_j)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{1}{m} \cdot y_j^2 = \|y\|_2^2.$$

Restriction to Unit Ball

Want to show that with high probability, $\|\mathbf{S}y\|_2 \approx_\epsilon \|y\|_2$ for all $\underbrace{y \in \{Ax : x \in \mathbb{R}^d\}}_{\text{col}(A)}$. I.e., for all $y \in \mathcal{V}$, where $\mathcal{V}$ is A's column span.

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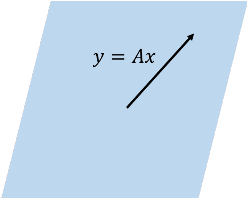
Observation: Suffices to prove $\|\mathbf{S}y\|_2 \approx_\epsilon \|y\|_2 = 1$ for all $y \in S_\mathcal{V}$ where

$$S_\mathcal{V} = \{y \,:\, y \in \mathcal{V} \text{ and } \|y\|_2 = 1\}.$$



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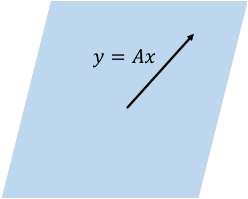

Proof: For any $y \in \mathcal{V}$, can write $y = \|y\|_2 \cdot \bar{y}$ where $\bar{y} = y/\|y\|_2 \in S_{\mathcal{V}}$.

$$(1-\epsilon) \leq \|\mathbf{S}\bar{y}\|_2 \leq (1+\epsilon) \implies$$

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Discretization of Unit Ball

Theorem

For any $\epsilon \leq 1$, there exists a set of points $\mathcal{N}_\epsilon \subset S_\mathcal{V}$ with $|\mathcal{N}_\epsilon| = \left(\frac{4}{\epsilon}\right)^d$ such that, for all $y \in S_\mathcal{V}$,

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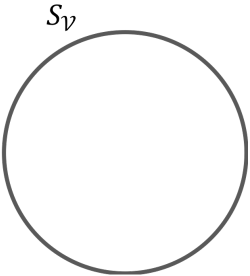

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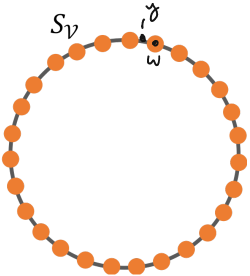
"ε-Net"



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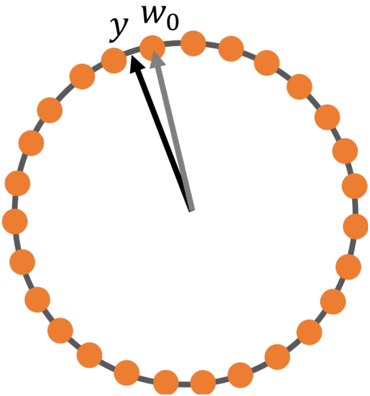

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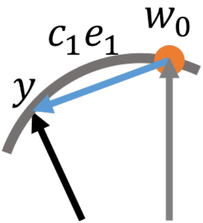

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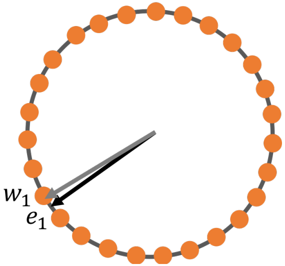

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For all i, have $c_i \leq \epsilon^i$.

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Have written $y \in S_{\mathcal{V}}$ as $y = w_0 + c_1 w_1 + c_2 w_2 + \dots$ where $w_0, w_1, \dots \in \mathcal{N}_\epsilon$, and $c_i \leq \epsilon^i$.

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$$\|\mathbf{S}y\|_2 = \|\mathbf{S}w_0 + c_1\mathbf{S}w_1 + c_2\mathbf{S}w_2 + \ldots\|_2$$

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(since via the union bound, $\|\mathbf{S}w\|_2 \approx \|w\|_2$ for all $w \in \mathcal{N}_\epsilon$)

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Similarly, can prove that $\|\mathbf{S}y\|_2 \geq 1 - 2\epsilon$, giving, for all $y \in S_{\mathcal{V}}$ (and hence all $y \in \mathcal{V}$):

$$(1-2\epsilon)\|y\|_2 \leq \|\mathbf{S}y\|_2 \leq (1+2\epsilon)\|y\|_2.$$

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- There exists an ϵ-net $\mathcal{N}_\epsilon$ over the unit ball in A's column span, $S_\mathcal{V}$ with $|\mathcal{N}_\epsilon| \leq \left(\frac{4}{\epsilon}\right)^d$.

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$\implies$ $\mathsf{S} \in \mathbb{R}^{m\times n}$ is an ϵ-subspace embedding for A.

Net Construction

Theorem (ϵ-net over ℓ_2 ball)

For any $\epsilon \leq 1$, there exists a set of points $\mathcal{N}_\epsilon \subset S_\mathcal{V}$ with $|\mathcal{N}_\epsilon| = \left(\frac{4}{\epsilon}\right)^d$ such that, for all $y \in S_\mathcal{V}$,

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Theoretical algorithm for constructing $\mathcal{N}_\epsilon$:

- Initialize $\mathcal{N}_\epsilon = \{\}$.
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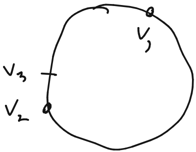

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If the algorithm terminates in T steps, we have $|\mathcal{N}_\epsilon| \leq T$ and $\mathcal{N}_\epsilon$ is a valid ϵ-net.

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How large is the net constructed by our theoretical algorithm?

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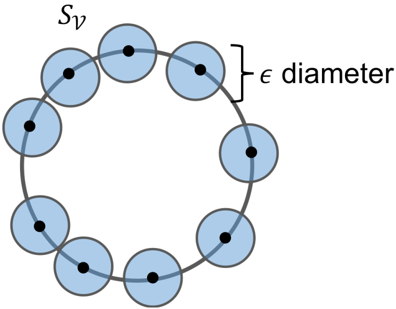
Thus, we can place an $\epsilon/2$ radius ball around each $w \in \mathcal{N}_\epsilon$, and none of these balls will intersect.



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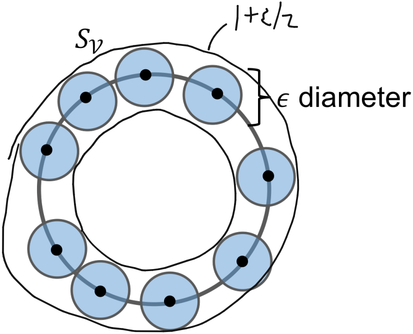
orange picking



Note that all these balls lie within the ball of radius $(1 + \epsilon/2)$.

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Remark: We never actually construct an ϵ-net. We just use the fact that one exists (the output of this theoretical algorithm) in our subspace embedding proof.