COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Spring 2026.

Lecture 4

Logistics

- Problem Set 1 due next Friday 2/20, at 11:59pm.
- For the Wikipedia Mark-and-Recapture question I posted a file with random URLs in case you are having trouble with requests to Wikipedia:Random.
- I am out of town next week. So Tuesday's lecture will be over Zoom. Link will be posted in Piazza. No lecture Thursday (Monday schedule for UMass).
- I will hold my usual Tuesday 2:30pm office hours, over the same Zoom link.

Last Time

Last Class:

- 2-level hashing and its analysis via linearity of expectation. Gives optimal $O(1)$ query time and $O(m)$ expected space usage.
- Practical random hash functions: 2-universal and pairwise independent hashing. $h(x) = ax + b \mod p$

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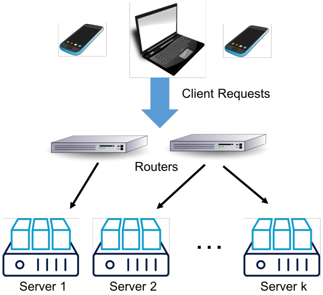
This Time:

- Hashing for load balancing in distributed systems. Motivating:
  - Stronger concentration inequalities: Chebyshev's inequality, exponential tail bounds, and their connections to the law of large numbers and central limit theorem.
  - The union bound to bound the probability that one of multiple possible correlated events happens.
- Some of the problem set questions use Chebyshev's inequality. After today you will be able to solve them.

Another Application

Randomized Load Balancing:



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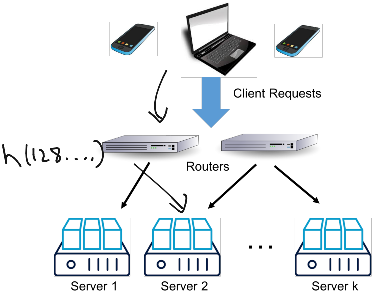

Simple Model: *n* requests randomly assigned to *k* servers. How many requests must each server handle?

- Often assignment is done via a random hash function. Why? Why not just a random number generator?

cache locality

Weakness of Markov's

R_i = #requests assigned to server i. n requests

k servers

$$\mathbb{E}[R_i] = \frac{n}{k}$$

n: total number of requests, k: number of servers randomly assigned requests, R_i: number of requests assigned to server i.

Weakness of Markov's

$$\mathbb{E}[\mathsf{R}_i] = \sum_{j=1}^{n} \mathbb{E}[\mathbb{I}_{\text{request } j \text{ assigned to } i}] = \sum_{j=1}^{n} \Pr\left[j \text{ assigned to } i\right] = \frac{n}{k}.$$

$$\frac{1}{k}$$

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If we provision each server be able to handle twice the expected load, what is the probability that a server is overloaded? → Markov's inequality

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Applying Markov's Inequality

$$\Pr[R_i \geq 2\mathbb{E}[R_i]] \leq \frac{\mathbb{E}[R_i]}{2\mathbb{E}[R_i]} = \frac{1}{2}.$$

server capacity

> n: total number of requests, k: number of servers randomly assigned requests, R_i: number of requests assigned to server i.

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$$\Pr[R_i \geq 2\mathbb{E}[R_i]] \leq \frac{\mathbb{E}[R_i]}{2\mathbb{E}[R_i]} = \frac{1}{2}.$$

Not great...half the servers may be overloaded.

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Chebyshev's inequality

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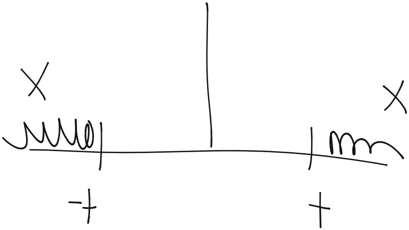
$$\Pr(|X| \geq t) = \Pr(X^2 \geq t^2).$$



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$X - \mathbb{E}X$

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Chebyshev's inequality: $\mathbb{E}[(X-\mathbb{E}X)^2]$

$$\Pr(|X-\mathbb{E}X|^2 \geq t^2) = \Pr(|X - \mathbb{E}[X]| \geq t) \leq \frac{\text{Var}[X]}{t^2}.$$

(by plugging in the random variable $X - \mathbb{E}[X]$)

Chebyshev's inequality

$$\Pr(|X - \mathbb{E}[X]| \geq t) \leq \frac{\mathrm{Var}[X]}{t^2}$$

X: any random variable, t, s: any fixed numbers.

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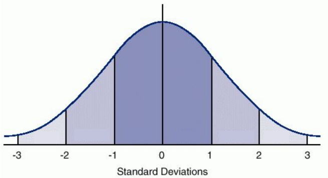
What is the probability that X falls s standard deviations from it's mean?



$$t = s\sqrt{Var(X)}$$

$$\Pr\left(|X - \mathbb{E}X| \geq s \cdot \sqrt{Var(X)}\right) \leq \frac{Var(X)}{s^2 \cdot Var(X)} = \frac{1}{s^2}$$

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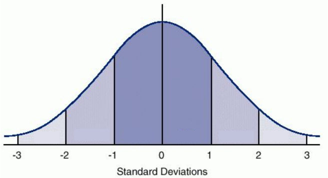

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Law of Large Numbers

Consider drawing independent identically distributed (i.i.d.) random variables $X_1, \ldots, X_n$ with mean μ and variance σ^2.

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$$= \frac{1}{n^2} \cdot \sum_{i=1}^{n} \text{Var}(X_i)$$

$$\frac{1}{n^2} \cdot n \cdot \sigma^2 = \boxed{\frac{\sigma^2}{n}}$$

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By Chebyshev's Inequality: for any fixed value $\epsilon > 0$,

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Law of Large Numbers: with enough samples n, the sample average will always concentrate to the mean.

$n \to \infty$ $\quad$ $\Pr(|S-\mu| \geq \varepsilon) \to 0$ for any fixed ε.

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- Cannot show from vanilla Markov's inequality.

Load Balancing Variance

We can write the number of requests assigned to server i, R_i as:

$\mathbb{E}R_i = \frac{n}{k}$

$Var(R_i) = ?$

$$R_i = \sum_{j=1}^{n} R_{i,j}$$

where $R_{i,j}$ is 1 if request j is assigned to server i and 0 otherwise.

n: total number of requests, k: number of servers randomly assigned requests, R_i: number of requests assigned to server i.

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We can write the number of requests assigned to server i, $\mathbf{R}_i$ as:

$$\text{Var}[\mathbf{R}_i] = \sum_{j=1}^{n} \text{Var}[\mathbf{R}_{i,j}] \qquad \text{(linearity of variance)}$$

$p(1-p)$

where $\mathbf{R}_{i,j}$ is 1 if request j is assigned to server i and 0 otherwise.

$$\text{Var}(R_{ij}) = \mathbb{E}[(R_{ij} - \mathbb{E}R_{ij})^2] = \mathbb{E}[R_{ij}^2] - [\mathbb{E}R_{ij}]^2$$

$$= \frac{1}{k} - \frac{1}{k^2} = \frac{1}{k}\left(1 - \frac{1}{k}\right)$$

R_{ij}^2: 1 if $R_{ij} = 1$ w.p. $\frac{1}{k}$; 0 if $R_{ij} = 0$ w.p. $1 - \frac{1}{k}$

> n: total number of requests, k: number of servers randomly assigned requests, $\mathbf{R}_i$: number of requests assigned to server i.

$$\mathbb{E}R_{ij}^2 = \frac{1}{k} \cdot 1 + \left(1 - \frac{1}{k}\right) \cdot 0 = \frac{1}{k}$$

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$$\text{Var}[\mathsf{R}_{i,j}] = \mathbb{E}[\mathsf{R}_{i,j}^2] - \mathbb{E}[\mathsf{R}_{i,j}]^2$$

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Bounding the Load via Chebyshevs

Letting R_i be the number of requests sent to server i, $\mathbb{E}[\mathsf{R}_i] = \frac{n}{k}$ and $\text{Var}[\mathsf{R}_i] \leq \frac{n}{k}$.

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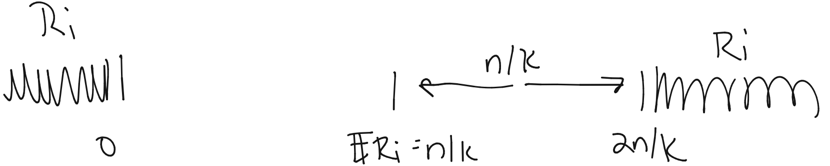
Applying Chebyshev's:

$$\Pr\left(R_i \geq \frac{2n}{k}\right) \leq \Pr\left(|R_i - \mathbb{E}[R_i]| \geq \frac{n}{k}\right) \leq \frac{\text{Var}(R_i)}{\left(\frac{n}{k}\right)^2} = \frac{\frac{n}{k}}{\left(\frac{n}{k}\right)^2} = \frac{k}{n}$$

server capacity = twice expected load



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$$\Pr\left(\mathsf{R}_i \ge \frac{2n}{k}\right) \le \Pr\left(|\mathsf{R}_i - \mathbb{E}[\mathsf{R}_i]| \ge \frac{n}{k}\right) \le \frac{n/k}{n^2/k^2}$$

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- Overload probability is small when $k \ll n$!
- Might seem counterintuitive – bound gets worse as k grows.
- When k is large, the number of requests each server sees in expectation is very small so the law of large numbers doesn't 'kick in'.

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Maximum Server Load

What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[R_i] = \frac{2n}{k}$. I.e., that some server is overloaded if we give each $\frac{2n}{k}$ capacity?

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$$\Pr\left(\max_i(\mathrm{R}_i) \geq \frac{2n}{k}\right)$$

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$$\Pr\left(\max_i(R_i) \geq \frac{2n}{k}\right) = \Pr\left(\left[R_1 \geq \frac{2n}{k}\right] \cup \left[R_2 \geq \frac{2n}{k}\right] \cup \ldots \cup \left[R_k \geq \frac{2n}{k}\right]\right)$$

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$$\Pr\left(\max_i(\mathsf{R}_i) \geq \frac{2n}{k}\right) = \Pr\left(\bigcup_{i=1}^{k}\left[\mathsf{R}_i \geq \frac{2n}{k}\right]\right)$$

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small $\leq \frac{k}{n}$

We want to show that $\Pr\left(\bigcup_{i=1}^{k}\left[R_i \geq \frac{2n}{k}\right]\right)$ is small.

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How do we do this? Note that $R_1, \ldots, R_k$ are correlated in a somewhat complex way.

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The Union Bound

Union Bound: For any random events $A_1, A_2, ..., A_k$,

$$\Pr\left(A_1 \cup A_2 \cup \ldots \cup A_k\right) \leq \Pr(A_1) + \Pr(A_2) + \ldots + Pr(A_k).$$

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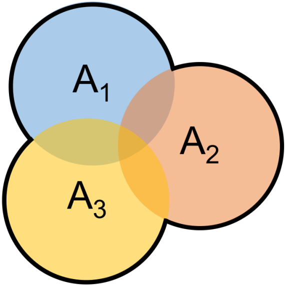

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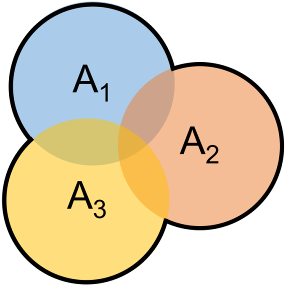
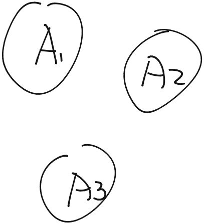
=?

special case of Markov's.





When is the union bound tight? — when disjoint

↳ nearly tight when independent with small probability.

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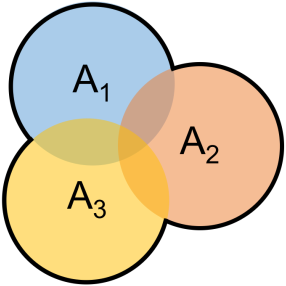

When is the union bound tight? When $A_1, ..., A_k$ are all disjoint.

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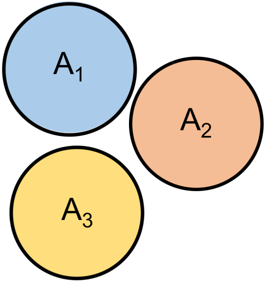

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Applying the Union Bound

What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[R_i] = \frac{2n}{k}$. I.e., that some server is overloaded if we give each $\frac{2n}{k}$ capacity?

$$\Pr\left(\max_i(R_i) \geq \frac{2n}{k}\right) = \Pr\left(\bigcup_{i=1}^{k}\left[R_i \geq \frac{2n}{k}\right]\right)$$

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Applying the Union Bound

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$$\leq \sum_{i=1}^{k} \Pr\left(\left[\mathrm{R}_i \geq \frac{2n}{k}\right]\right) \qquad \text{(Union Bound)}$$

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As long as $k \leq O(\sqrt{n})$, with good probability, the maximum server load will be small (compared to the expected load).

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Questions on union bound, Chebyshev's inequality, random hashing?

Flipping Coins

We flip $n = 100$ independent coins, each are heads with probability 1/2 and tails with probability 1/2. Let H be the number of heads.

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$$\mathbb{E}[H] = \frac{n}{2} = 50 \text{ and } \mathrm{Var}[H] = 25$$

$$\left(\sum_{i=1}^{100} Var(H_i) = \sum_{i=1}^{100} \frac{1}{2} - \frac{1}{2^2} \right.$$

$$= 100 \cdot \frac{1}{4} = 25.$$

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Markov's: $\frac{\mathbb{E}[H]}{60} = \frac{50}{60}$

$\Pr(H \geq 60) \leq .833$

$\Pr(H \geq 70) \leq .714$

$\Pr(H \geq 80) \leq .625$

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Markov's:	Chebyshev's:	In Reality:
$\Pr(\mathsf{H} \geq 60) \leq .833$	$\Pr(\mathsf{H} \geq 60) \leq .25$	$\Pr(\mathsf{H} \geq 60) = 0.0284$
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$\Pr(\mathsf{H} \geq 80) \leq .625$	$\Pr(\mathsf{H} \geq 80) \leq .0278$	$\Pr(\mathsf{H} \geq 80) < 10^{-9}$

H has a simple Binomial distribution, so can compute these probabilities exactly.

Tighter Concentration Bounds

To be fair.... Markov and Chebyshev's inequalities apply much more generally than to Binomial random variables like coin flips.

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- Chebyshev's: $\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr(|X - \mathbb{E}[X]|^2 \geq t^2) \leq \frac{\mathrm{Var}[X]}{t^2}$. Second Moment.
- What if we just apply Markov's inequality to even higher moments?

$$\mathbb{E}(|X - \mathbb{E}X|^4)$$

A Fourth Moment Bound

$$\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr\left((X - \mathbb{E}[X])^4 \geq t^4\right)$$

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$$\frac{Var(X)}{t^2}$$

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- Bound the fourth moment:

$$\mathbb{E}\left[(H - \mathbb{E}[H])^4\right] = \mathbb{E}\left[\left(\sum_{i=1}^{100} H_i - 50\right)^4\right]$$

where $H_i = 1$ if coin flip i is heads and 0 otherwise.

A Fourth Moment Bound

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- Apply Fourth Moment Bound: $\Pr(|H - \mathbb{E}[H]| \geq t) \leq \frac{1862.5}{t^4}$.

Tighter Bounds

Chebyshev's:	In Reality:
$\Pr(H \geq 60) \leq .25$	$\Pr(H \geq 60) = 0.0284$
$\Pr(H \geq 70) \leq .0625$	$\Pr(H \geq 70) = .000039$
$\Pr(H \geq 80) \leq .04$	$\Pr(H \geq 80) < 10^{-9}$

H: total number heads in 100 random coin flips. $\mathbb{E}[H] = 50$.

Tighter Bounds

Chebyshev's:	4^{th} Moment:	In Reality:
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Can we just keep applying Markov's inequality to higher and higher moments and getting tighter bounds?

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$\Pr(|H - \mathbb{E}H| \geq t) = \Pr(f(|H - \mathbb{E}H|) \geq f(t))$

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Next time: Use this approach to give exponential tail bounds, and a quantitative understanding of the central limit theorem.

- Leads to much better bounds for random hash tables, randomized load balancing, and may other randomized algorithms.

$$(H - \mathbb{E}H)^4 = \left[\left(\sum_{i=1}^{100} H_i\right) - 50\right]^4$$