# COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Cameron Musco

University of Massachusetts Amherst. Spring 2020.

Lecture 8

- Problem Set 1 was due this past Friday. Will be graded by next week.
- Problem Set 2 to be released end of this week and due $\sim$ 3/6.

- We take academic honestly on the problem sets seriously.

# ACADEMIC HONESTY ON PROBLEM SETS

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- If caught copying from another group (or allowing someone to copy your work), copying from problem sets or answer keys from past semesters, etc. you will receive a 0**% on the problem set and** 5**% off your final course grade.**

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- You can change your problem set group from assignment to assignment.

Last Class:

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- SimHash for cosine similarity
- Applications to e.g., approximate neural network computation.
- Introduction to the Frequent Elements (heavy-hitters) problem in data streams.
- The Boyer-Moore voting algorithm for majority.

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**This Class:**

- Extend Boyer-Moore to the general Frequent Elements: problem: Misra-Gries summaries.
- Count-min sketch (random hashing for frequent element estimation).

**Next Few Classes:**

- Random compression methods for high dimensional vectors. The Johnson-Lindenstrauss lemma.
- Compressed sensing (sparse recovery) and connections to the frequent elements problem.

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- Spectral clustering and spectral graph theory.

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- PCA, low-rank approximation, and the singular value decomposition.
- Spectral clustering and spectral graph theory.

Will use a lot of linear algebra. May be helpful to refresh.

- Vector dot product, addition, length. Matrix vector multiplication.
- Linear independence, column span, orthogonal bases, rank.
- Orthogonal projection, eigendecomposition, linear systems.

**$k$-Frequent Items (Heavy-Hitters) Problem**: Consider a stream of $n$ items $x_1, \ldots, x_n$ (with possible duplicates). Return any item that appears at least $\frac{n}{k}$ times. E.g., for $n = 9$, $k = 3$:

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ |
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- Think of $k = 100$. Want items appearing $\geq 1\%$ of the time.
- Easy with $O(n)$ space – store the count for each item and return the one that appears $\geq n/k$ times.

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**Applications:** Finding viral products/media/searches, frequent itemset mining, detecting DoS and other attacks, 'iceberg queries' in databases.

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**Boyer-Moore Voting Algorithm:**

k=2

- Initialize count $c := 0$, majority element $m := \perp$
- For $i = 1, \ldots, n$
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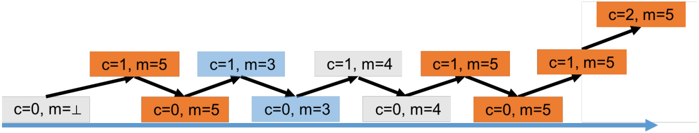


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# MISRA-GRIES SUMMARIES

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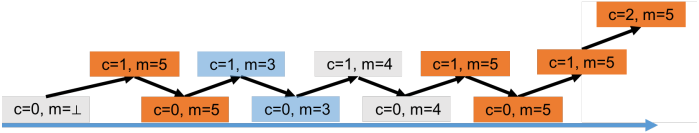


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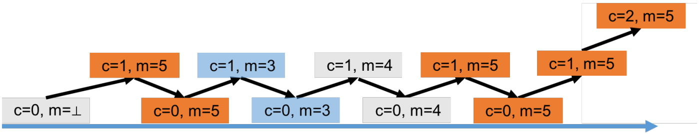


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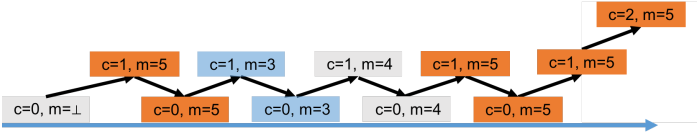


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$c_1$=1, $m_1$=5

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$c_2=1, m_2=12$

$c_3=0, m_3=\perp$

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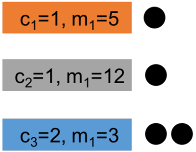


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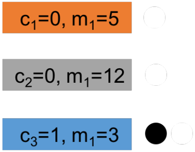


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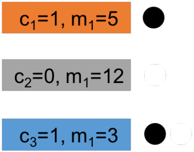


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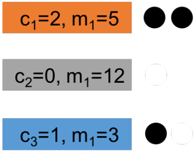


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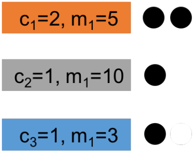


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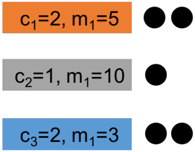


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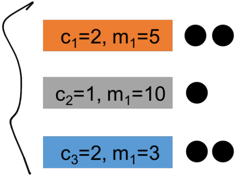
$\geq \frac{9}{3} = \frac{9}{3} = 3$

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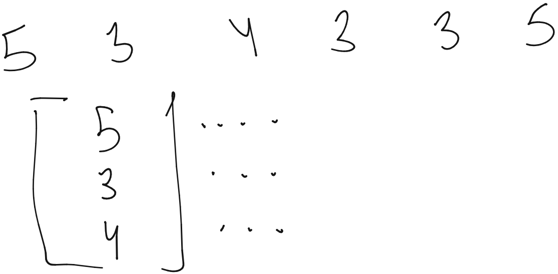

# MISRA-GRIES ANALYSIS

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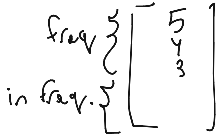
- If there are $k/2$ items each appearing $\geq n/k$ times, there are $\leq n/2$ irrelevant items, being inserted into $k/2$ 'free slots'.

5 1 5 2 5 11 10 3 4 3 12 3

$$\frac{k}{2} \cdot \frac{n}{k} = \frac{n}{2}$$



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Anything undesirable about the Misra-Gries output guarantee?

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Anything undesirable about the Misra-Gries output guarantee?

May have false positives – infrequent items that are stored.

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| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | ... | $x_{n-n/k+1}$ | ... | $x_n$ |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 12 | 9 | 27 | 4 | 101 | | 3 | | 3 |

n/k-1 occurrences

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$(\epsilon, k)$**-Frequent Items Problem**: Consider a stream of $n$ items $x_1, \ldots, x_n$. Return a set $F$ of items, including all items that appear at least $\frac{n}{k}$ times and only items that appear at least $(1-\epsilon) \cdot \frac{n}{k}$ times.

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- An example of relaxing to a 'promise problem': for items with frequencies in $[(1-\epsilon) \cdot \frac{n}{k}, \frac{n}{k}]$ no output guarantee.

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- Let $r := \lceil k/\epsilon \rceil$
- Initialize counts $c_1, \ldots, c_r := 0$, elements $m_1, \ldots, m_r := \perp$.
- For $i = 1, \ldots, n$
  - If $m_j = x_i$ for some $j$, set $c_j := c_j + 1$.
  - Else let $t = \arg\min c_j$. If $c_t = 0$, set $m_t := x_i$ and $c_t := 1$.
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**Implication:** If $f(m_j) \geq \frac{n}{k}$, then $c_j \geq (1 - \epsilon) \cdot \frac{n}{k}$ so the item is returned.
If $f(m_j) < (1 - \epsilon) \cdot \frac{n}{k}$, then $c_j < (1 - \epsilon) \cdot \frac{n}{k}$ so the item is not returned.

**Upshot:** The $(\epsilon, k)$-Frequent Items problem can be solved via the Misra-Gries approach.

$$\left[(1-\varepsilon)\left(\frac{n}{k}\right), \frac{n}{k}\right]$$

**Upshot:** The $(\epsilon, k)$-Frequent Items problem can be solved via the Misra-Gries approach.

- Space usage is ($r =$) $\lceil k/\epsilon \rceil$ counts – $O\left(\frac{k \log n}{\epsilon}\right)$ bits and $\lceil k/\epsilon \rceil$ items.
- Deterministic approximation algorithm.

Search if is item is stored $O(1)$

Perform decs. $\frac{k}{\varepsilon}$ time

# FREQUENT ELEMENTS WITH COUNT-MIN SKETCH

A common alternative to the Misra-Gries approach is the count-min sketch: a randomized method closely related to bloom filters.

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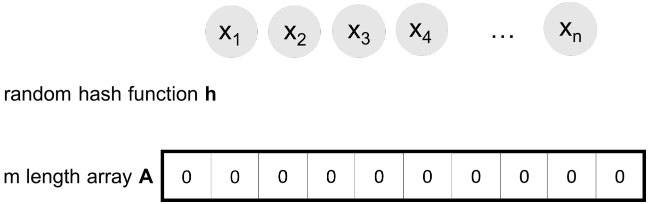

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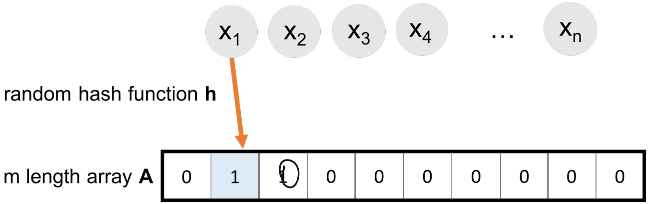

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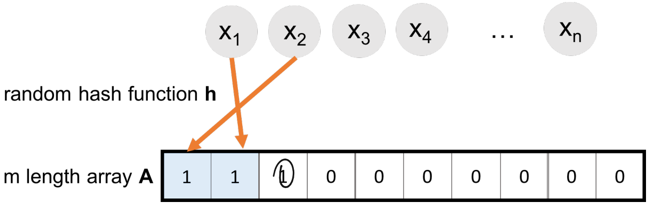

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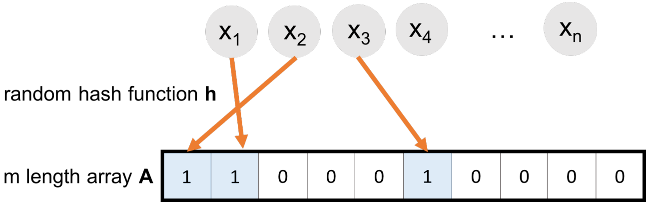

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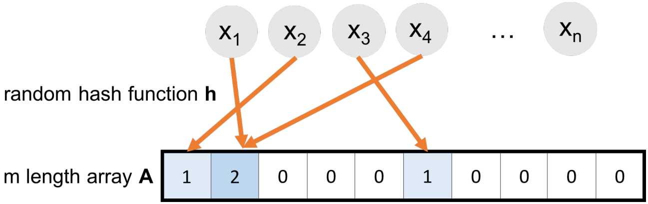

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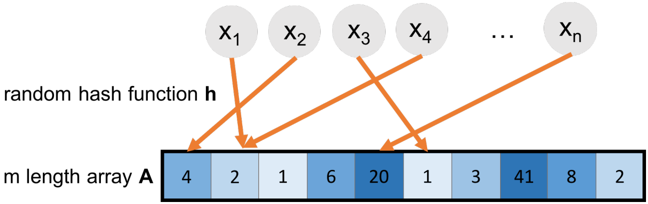

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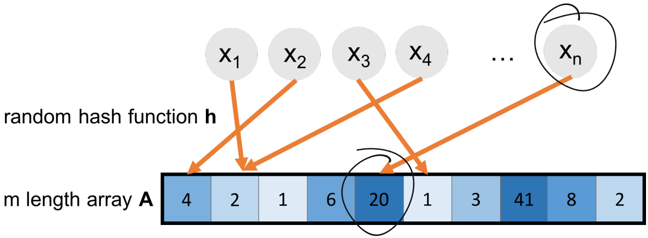


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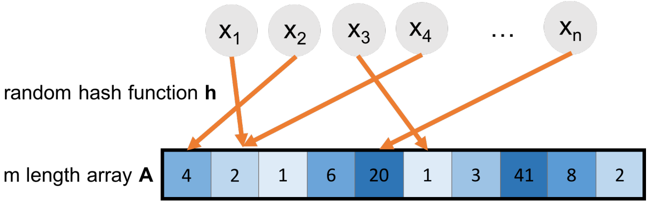


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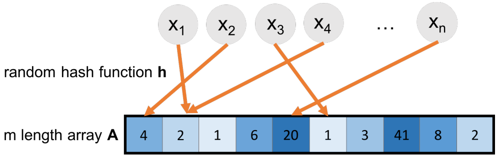
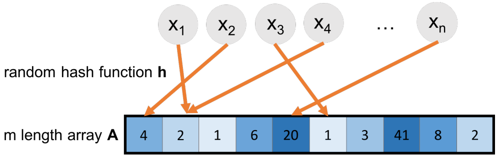
# COUNT-MIN SKETCH ACCURACY




Use $A[\mathbf{h}(x)]$ to estimate $f(x)$

$f(x)$: frequency of $x$ in the stream (i.e., number of items equal to $x$). $\mathbf{h}$: random hash function. $m$: size of count-min sketch array.

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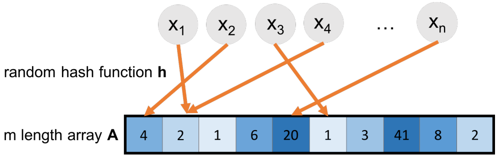


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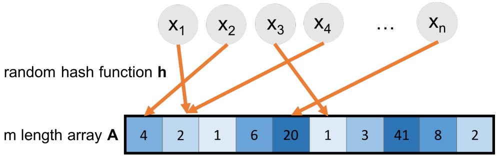


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- $A[\mathbf{h}(x)] = f(x) + \sum_{y \neq x : \mathbf{h}(y) = \mathbf{h}(x)} f(y)$.

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