# COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Cameron Musco
University of Massachusetts Amherst. Spring 2020.
Lecture 7

# LOGISTICS

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- Talk Today: Vatsal Sharan at 4pm in CS 151. *Modern Perspectives on Classical Learning Problems: Role of Memory and Data Amplification.*

Last Class:

## Last Class: Hashing for Jaccard Similarity

- MinHash for estimating the Jaccard similarity.
- Locality sensitive hashing (LSH).
- Application to fast similarity search.

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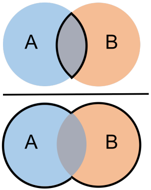
This Class:

- Finish up MinHash and LSH.
- The Frequent Elements (heavy-hitters) problem.
- Misra-Gries summaries.

**Jaccard Similarity:** $J(A,B) = \frac{|A \cap B|}{|A \cup B|} = \frac{\text{\# shared elements}}{\text{\# total elements}}$.




**Two Common Use Cases:**

- **Near Neighbor Search:** Have a database of $n$ sets/bit strings and given a set $A$, want to find if it has high similarity to anything in the database. Naively $\Omega(n)$ time.
- **All-pairs Similarity Search:** Have $n$ different sets/bit strings. Want to find all pairs with high similarity. Naively $\Omega(n^2)$ time.

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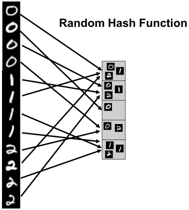


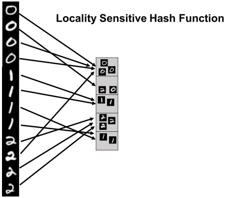

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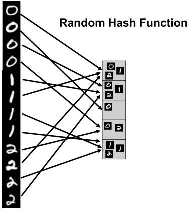


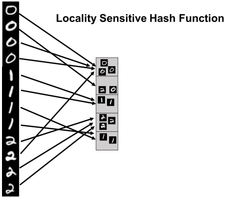


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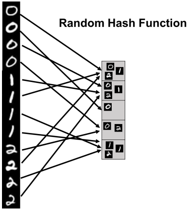
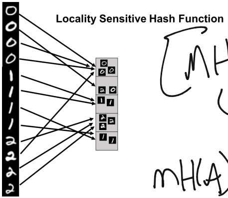


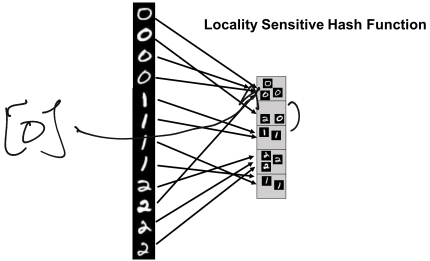




MH(A) = MH(B) → P = 1

MH(A) ≠ MH(B) < 1/m

What is $\Pr\left[\mathbf{g}(MinHash(A)) = \mathbf{g}(MinHash(B))\right]$ if $\mathbf{g}$ is not collision free?
Will be a bit larger than $J(A, B)$.

When searching for similar items only search for matches that land in the same hash bucket.

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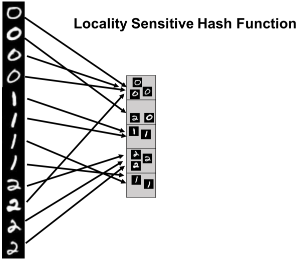


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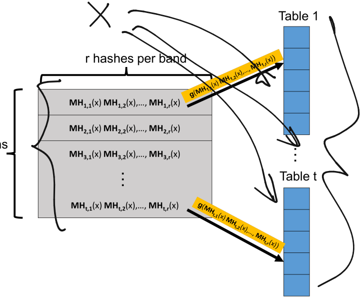
Need to balance a small probability of false negatives (a high hit rate) with a small probability of false positives (a small query time.)

Balancing False Negatives/Positives with MinHash via repetition.


Create $t$ hash tables. Each is indexed into not with a single MinHash value, but with $r$ values, appended together. A length $r$ signature:

$$\mathrm{MH}_{i,1}(x), \mathrm{MH}_{i,2}(x), \ldots, \mathrm{MH}_{i,r}(x).$$

**Hit Rate:** Given by the s-curve: $1-(1-s^r)^t$.

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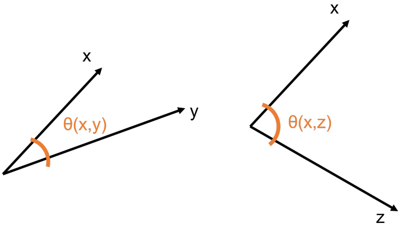

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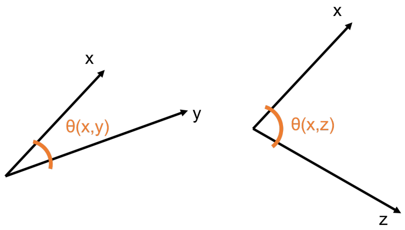


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# LOCALITY SENSITIVE HASHING

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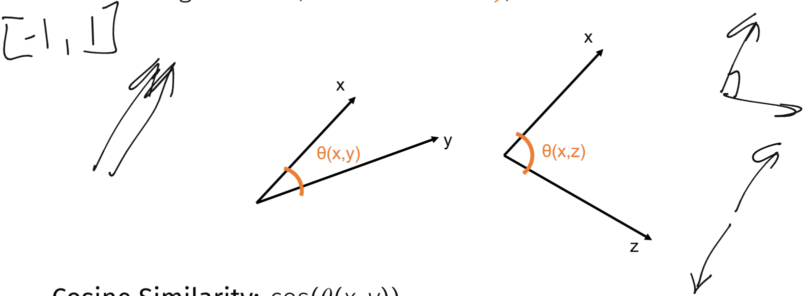


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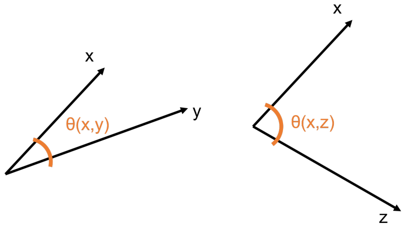


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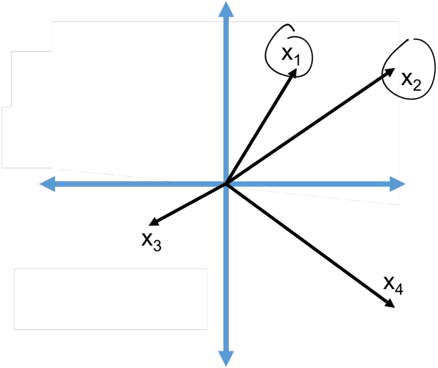

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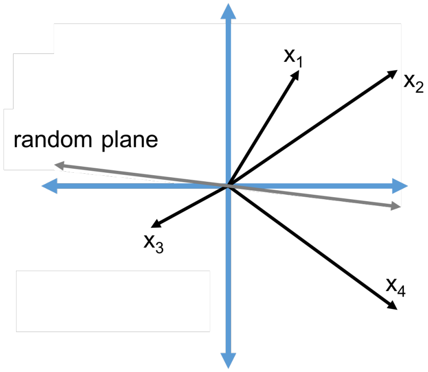

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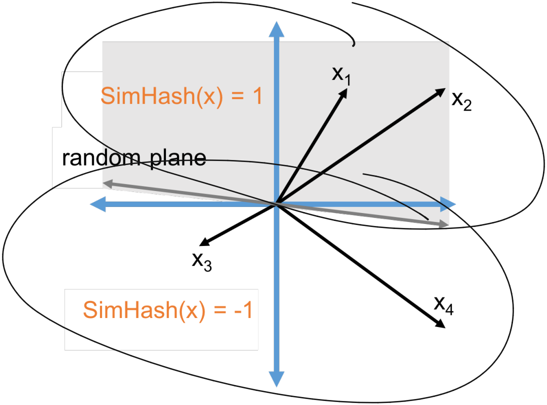

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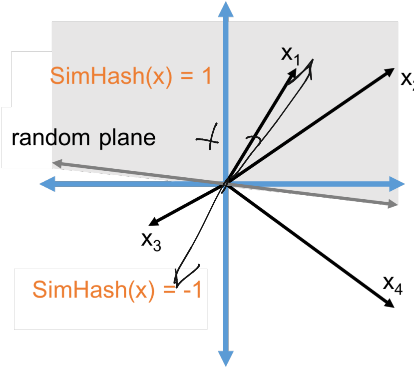


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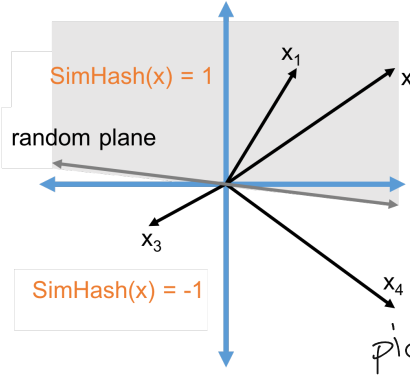


pick t once
use it in all calls to SH.

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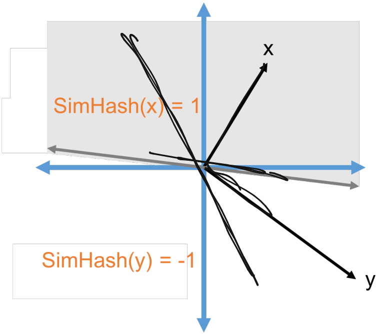

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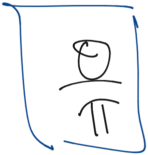

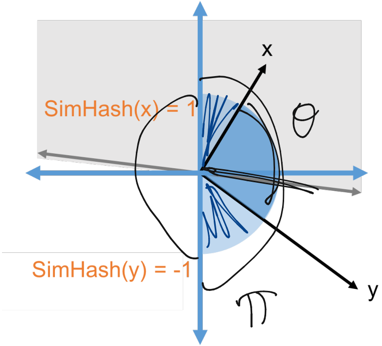


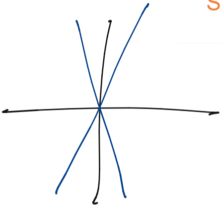

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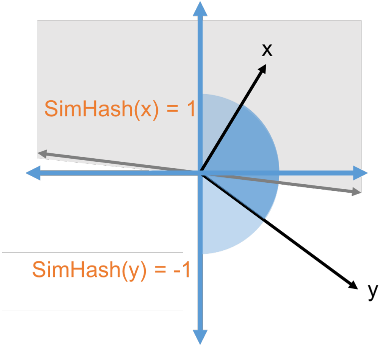


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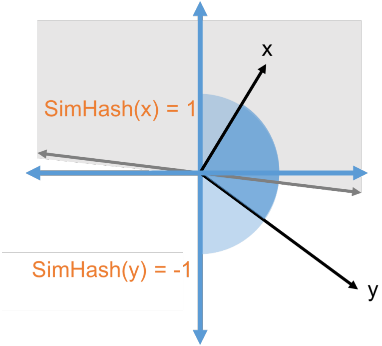
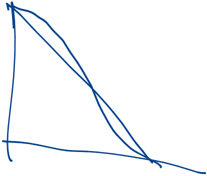
sign(<x,t>)

{-1,1}



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- $\Pr\left[SimHash(x) = SimHash(y)\right] = 1 - \frac{\theta(x,y)}{\pi} \approx \frac{\cos(\theta(x,y))+1}{2}$

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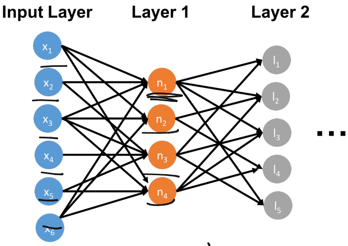


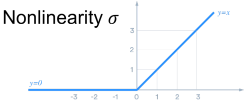


ReLU

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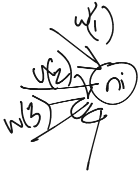

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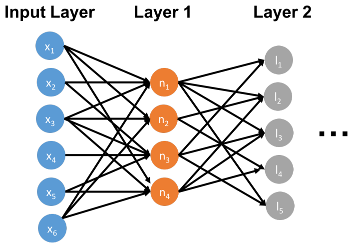


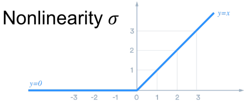


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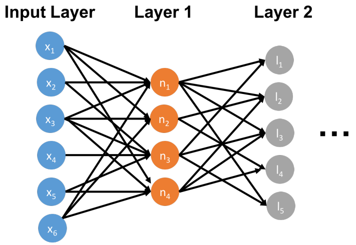


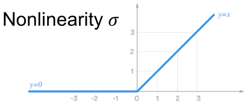


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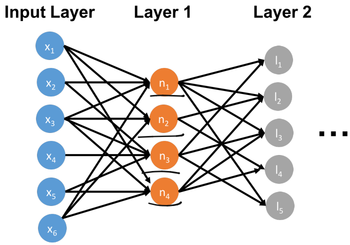


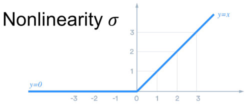


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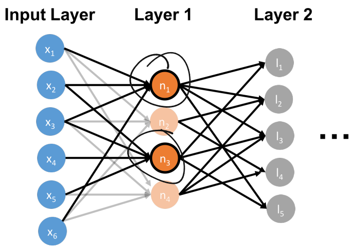


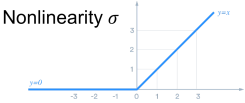


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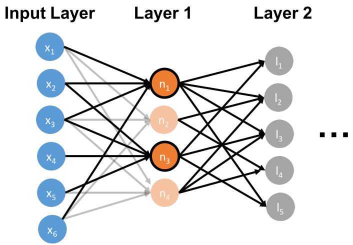


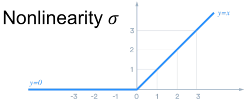


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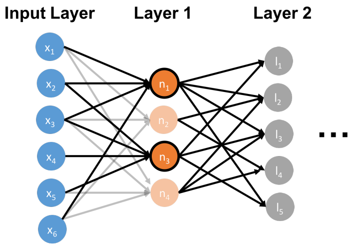


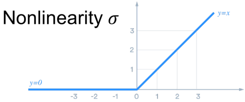


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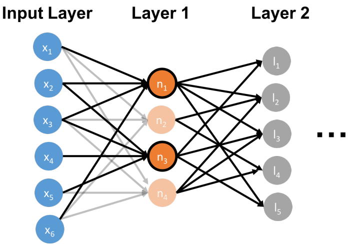
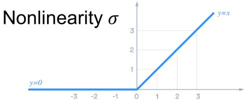
# HASHING FOR NEURAL NETWORKS

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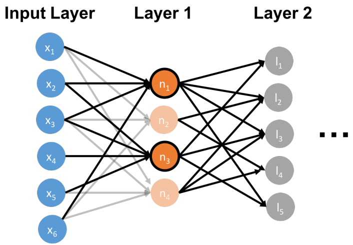
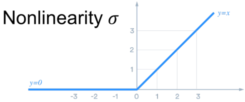
- Important neurons have high activation $\sigma(\langle w_i, x\rangle)$.
- Since $\sigma$ is typically monotonic, this means large $\langle w_i, x\rangle$.

x + wi are similar under cosine similarity




$$n_i = \sigma\left(\sum_{j=1}^{m} w(x_j, n_i)\cdot x_j\right) = \sigma(\langle w_i, x\rangle)$$

- Important neurons have high activation $\sigma(\langle w_i, x\rangle)$.
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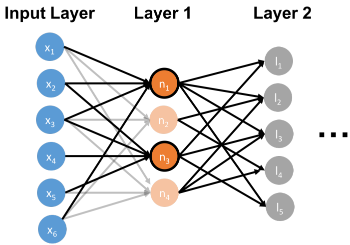
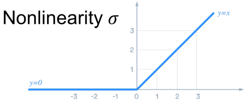
- $\cos(\theta(w_i, x)) = \frac{\langle w_i, x\rangle}{\|w_i\|\|x\|}$. Thus these neurons can be found very quickly using LSH for cosine similarity search.

# HASHING FOR NEURAL NETWORKS

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$SH(x) = \text{sign}(\langle x, t\rangle)$

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- $\cos(\theta(w_i, x)) = \frac{\langle w_i, x\rangle}{\|w_i\|\|x\|}$. Thus these neurons can be found very quickly using LSH for cosine similarity search.
- Store each weight vector $w_i$ (corresponding to each node) in a set of hash tables and check inputs $x$ for similarity to these stored vectors.

Questions on MinHash and Locality Sensitive Hashing?

$k$-Frequent Items (Heavy-Hitters) Problem: Consider a stream of $n$ items $x_1, \ldots, x_n$ (with possible duplicates). Return any item at appears at least $\frac{n}{k}$ times.

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|---|---|---|---|---|---|---|---|---|
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- Similar challenge as with the distinct elements problem.

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Generally want very fast detection, without having to scan through database/logs. I.e., want to maintain a running list of frequent items that appear in a stream.

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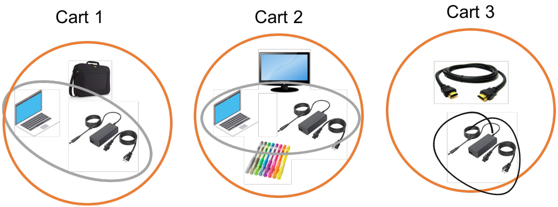

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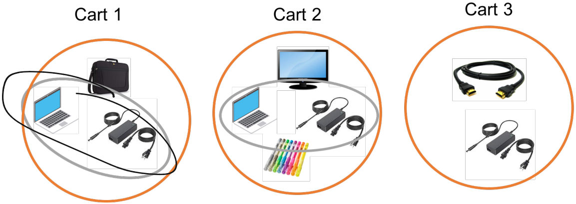


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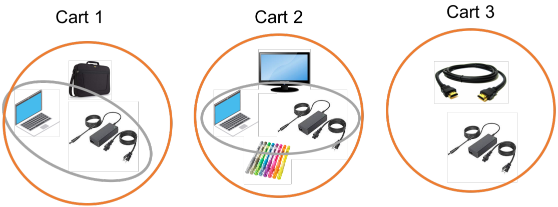


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- Frequency of an itemset is known as its support.
- A single basket includes many different itemsets, and with many different baskets an efficient approach is critical. E.g., baskets are Twitter users and itemsets are subsets of who they follow.

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- Basically $k$-Frequent items for $k = 2$ (and assume a single item has a strict majority.)

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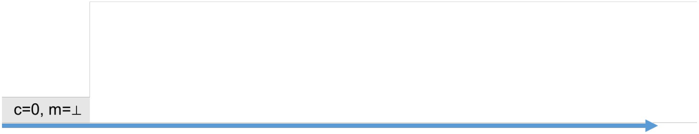


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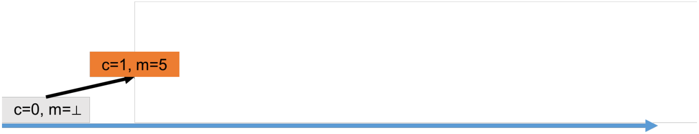


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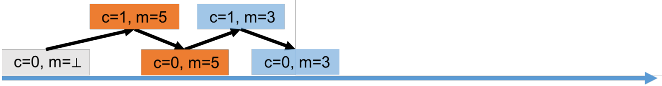


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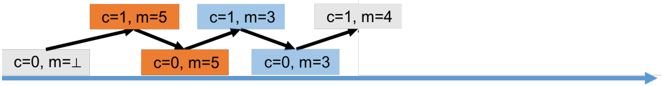


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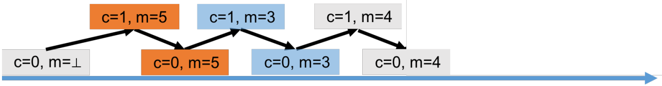


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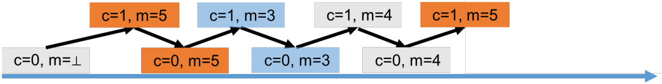


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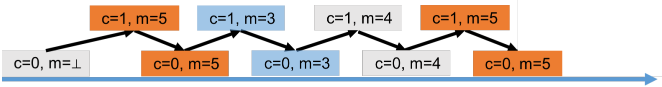


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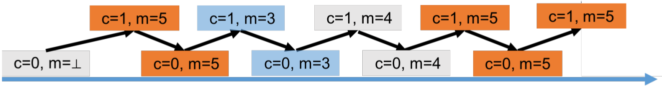


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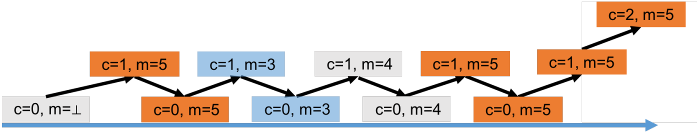


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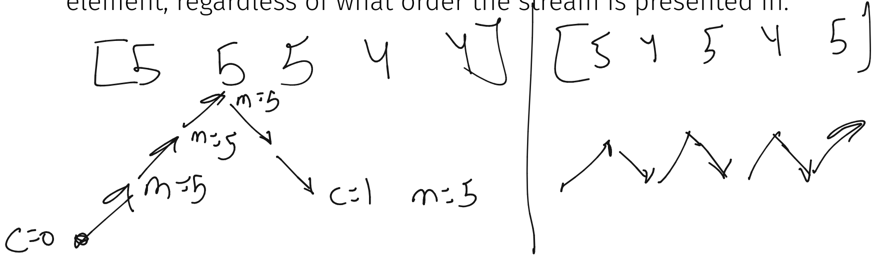

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**Proof:** Let $M$ be the true majority element. Let $s = c$ when $m = M$ and $s = -c$ otherwise

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**Claim:** The Boyer-Moore algorithm always outputs the majority element, regardless of what order the stream is presented in.

**Proof:** Let $M$ be the true majority element. Let $s = c$ when $m = M$ and $s = -c$ otherwise (s is a 'helper' variable).

# CORRECTNESS OF BOYER-MOORE

**Boyer-Moore Voting Algorithm:**

- Initialize count $c := 0$, majority element $m := \perp$
- For $i = 1, \ldots, n$
  - If $c = 0$, set $m := x_i$ and $c := 1$.
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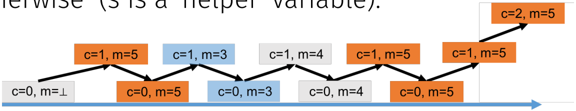


| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 12 | 3 | 5 | 4 | 5 | 5 | 10 | 5 | 5 |

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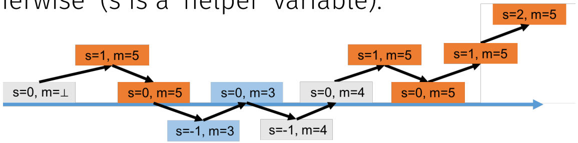


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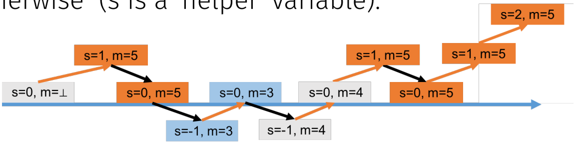


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- $s$ is incremented each time $M$ appears. So it is incremented more than it is decremented (since $M$ appears a majority of times) and ends at a positive value. $\implies$ algorithm ends with $m = M$.

**Next Time:** Will see a variant on the Boyer-Moore algorithm – the Misra-Greis summary.

- Stores $k$ top items at once and solves the Frequent Items problem.