COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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University of Massachusetts Amherst. Spring 2020.

Lecture 5

LOGISTICS

- Problem Set 1 was released last Thursday and is due Friday 2/14 at 8pm in Gradescope. Don't leave until the last minute.
- There is **no class** this Thursday.

Last Class We Covered:

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- Bloom Filters:
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- **Bloom Filters:**
  - Random hashing to maintain a large set in very small space.
  - Discussed applications and how the false positive rate is determined.
- **Streaming Algorithms and Distinct Elements:**
  - Started on streaming algorithms and one of the most fundamental examples: estimating the number of distinct items in a data stream.
  - Introduced an algorithm for doing this via a min-of-hashes approach.

Finish Distinct Elements:

- Finish hashing-based distinct elements algorithm. Learn the 'median trick' to boost accuracy.
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MinHashing For Set Similarity:

- See how a min-of-hashes approach (MinHash) is used to estimate the overlap between two bit vectors.
- A key idea behind audio fingerprint search (Shazam), document search (plagiarism and copyright violation detection), recommendation systems, etc.

Distinct Elements (Count-Distinct) Problem: Given a stream $x_1, \ldots, x_n$, estimate the number of distinct elements.

Hashing for Distinct Elements (variant of Flajolet-Martin):

- Let $\mathbf{h} : U \to [0, 1]$ be a random hash function (with a real valued output)
- $\mathbf{s} := 1$
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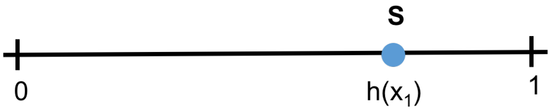

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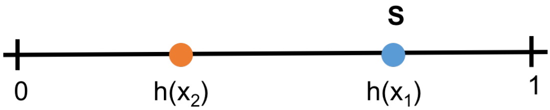

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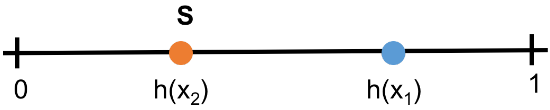

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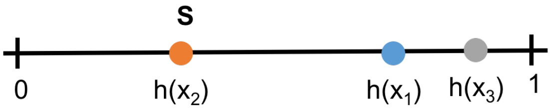

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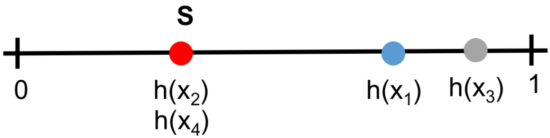

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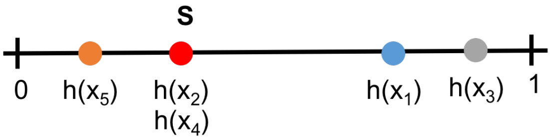

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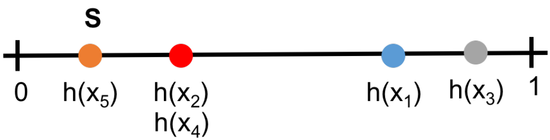

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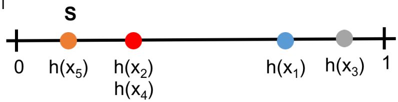

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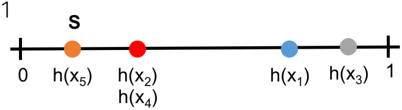

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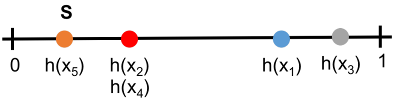

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- **Notice:** Output does not depend on n at all.

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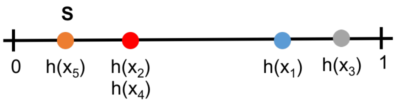

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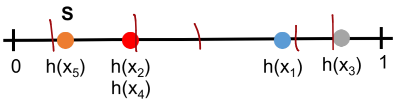

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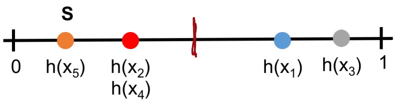

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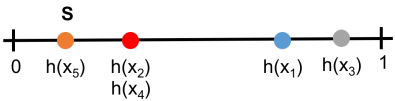

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$\tilde{d} = \frac{1}{1/d+1} - 1 = d$

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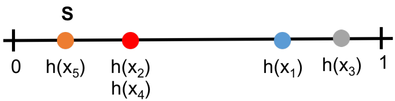

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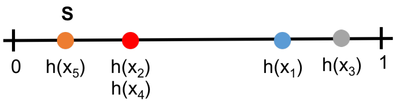

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PERFORMANCE IN EXPECTATION

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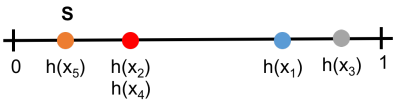

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- **Approximation is robust:** if $|\mathbf{s} - \mathbb{E}[\mathbf{s}]| \leq \epsilon \cdot \mathbb{E}[\mathbf{s}]$ for any $\epsilon \in (0, 1/2)$:

$$(1-2\epsilon)d \leq \widehat{\mathbf{d}} \leq (1+4\epsilon)d$$

.

So question is how well $\mathbf{s}$ concentrates around its mean.

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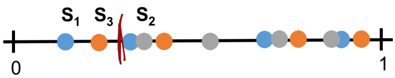

$|s - \mathbb{E}s| \leq \epsilon \mathbb{E}s$

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$$\mathbb{E}[s_j] = \frac{1}{d+1}$$

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s_j: minimum of d distinct hashes chosen randomly over $[0,1]$. $s = \frac{1}{k}\sum_{j=1}^{k} s_j$.
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ANALYSIS

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Hashing for Distinct Elements:

- Let $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_k : U \to [0, 1]$ be random hash functions
- $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k := 1$
- For $i = 1, \ldots, n$
  - For j=1,..., k, $\mathbf{s}_j := \min(\mathbf{s}_j, \mathbf{h}_j(x_i))$
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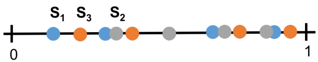

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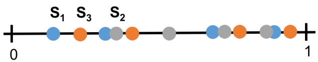

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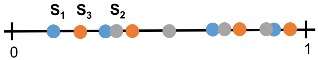

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  $\frac{1}{\delta} = \frac{1}{.05} = 20$ $\quad \frac{20}{\epsilon^2} \quad \frac{100}{\epsilon^2}$

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- $\delta = 5\%$ failure rate gives a factor 20 overhead in space complexity.

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$$\Pr\left(|\mathbf{X} - \mu| \geq t\right) \leq 2\exp\left(-\frac{t^2 k}{2\bar{\sigma}^2 + \frac{4}{3}\bar{M}t}\right).$$

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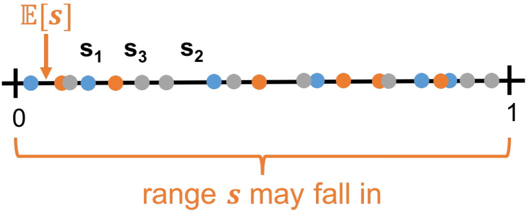
For us, $t = \frac{\epsilon}{d}$ and $\bar{M} = 1$. So $\frac{t^2 k}{\frac{4}{3}\bar{M}t} = \frac{3\epsilon k}{4d}$. So if $k \ll d$ exponent has small magnitude (i.e., bound is bad).

Exponential tail bounds are weak for random variables with very large ranges compared to their expectation.



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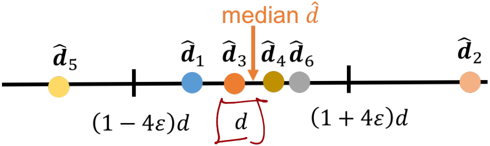

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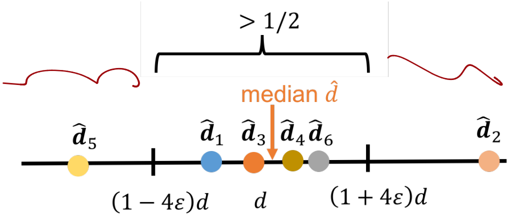

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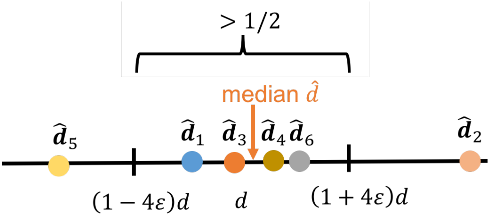

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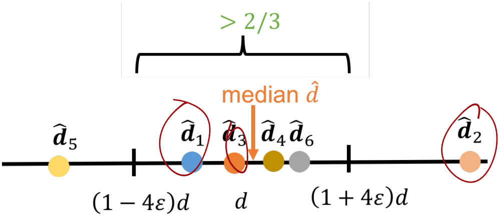

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- $\widehat{\mathsf{d}}_1, \ldots, \widehat{\mathsf{d}}_t$ are the outcomes of the t trials, each falling in $[(1-4\epsilon)d, (1+4\epsilon)d]$ with probability at least $4/5$.
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$$\Pr\left(\widehat{\mathbf{d}} \notin [(1-4\epsilon)d, (1+4\epsilon)d]\right) \leq \Pr\left(\mathbf{X} < \frac{5}{6} \cdot \mathbb{E}[\mathbf{X}]\right) \leq \Pr\left(|\mathbf{X} - \mathbb{E}[\mathbf{X}]| \geq \frac{1}{6}\mathbb{E}[\mathbf{X}]\right)$$

Apply Chernoff bound:

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THE MEDIAN TRICK

- $\widehat{\mathbf{d}}_1, \ldots, \widehat{\mathbf{d}}_t$ are the outcomes of the t trials, each falling in $[(1-4\epsilon)d, (1+4\epsilon)d]$ with probability at least $4/5$.
- $\widehat{\mathbf{d}} = median(\widehat{\mathbf{d}}_1, \ldots, \widehat{\mathbf{d}}_t)$.

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- Setting $t = O(\log(1/\delta))$ gives failure probability $e^{-\log(1/\delta)} = \delta$.

Upshot: The median of $t = O(\log(1/\delta))$ independent runs of the hashing algorithm for distinct elements returns $\widehat{\mathbf{d}} \in [(1-4\epsilon)d, (1+4\epsilon)d]$ with probability at least $1-\delta$.

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A note on the median: The median is often used as a robust alternative to the mean, when there are outliers (e.g., heavy tailed distributions, corrupted data).

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DISTINCT ELEMENTS IN PRACTICE

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Note: Careful averaging of estimates from multiple hash functions.

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Traditional *COUNT*, *DISTINCT* SQL calls are far too slow, especially when the data is distributed across many servers.

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IN PRACTICE

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- Count distinct keys where key is $(IP, Hr, Min \mod 10)$.
- Using HyperLogLog, cost is roughly that of a (distributed) linear scan (to stream through all items in table).

Questions on distinct elements counting?

Jaccard Index: A similarity measure between two sets.

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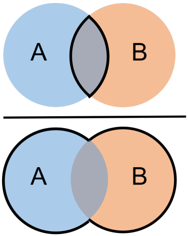

ANOTHER FUNDAMENTAL PROBLEM

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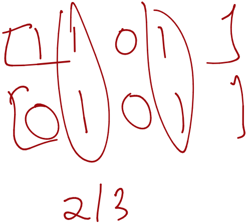

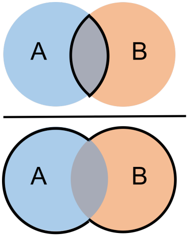

Natural measure for similarity between bit strings – interpret an n bit string as a set, containing the elements corresponding the positions of its ones. $J(x,y) = \frac{\#\text{ shared ones}}{\text{total ones}}$.

COMPUTING JACCARD SIMILARITY

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COMPUTING JACCARD SIMILARITY

Locality Sensitive Hashing

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Prohibitively expensive when n is very large. We'll see how to significantly improve on these runtimes with random hashing.

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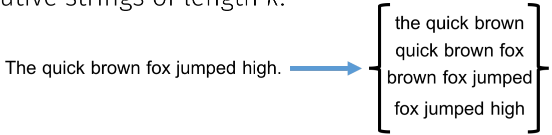

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E.g., to detect plagiarism and copyright infringement, to see if an email message is similar to previously seen spam, to detect duplicate webpages in search results, etc.

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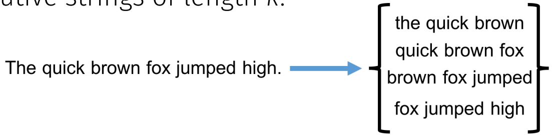

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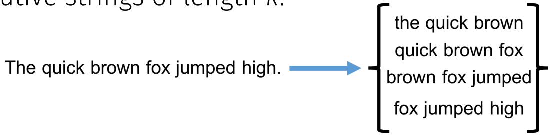

- Measure similarity as Jaccard similarity between shingle sets.
- Also used to measure word similarity. E.g., in spell checkers.

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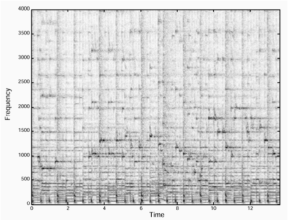
Step 1: Compute the spectrogram: representation of frequency intensity over time.



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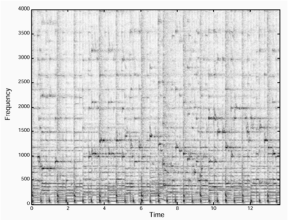

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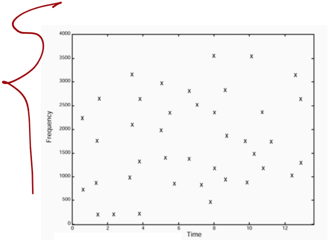

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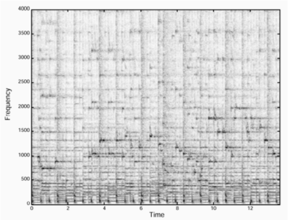

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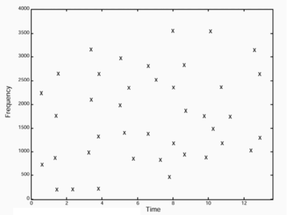
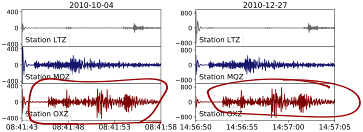


Compare thresholded spectrograms with Jaccard similarity.

Small earthquakes make consistent signatures on seismographs that repeat over time. Detecting repeated signatures lets you detect these otherwise undetectable events.



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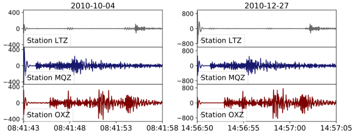

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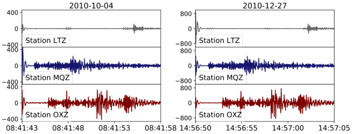

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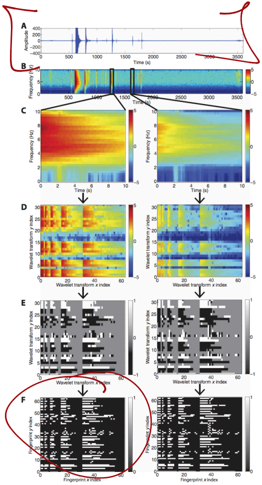

APPLICATION: EARTHQUAKE DETECTION

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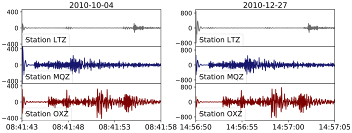
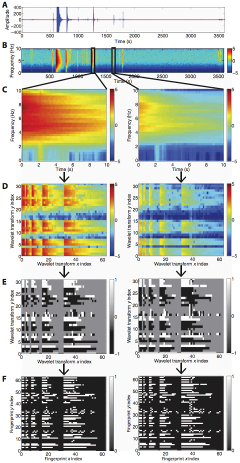
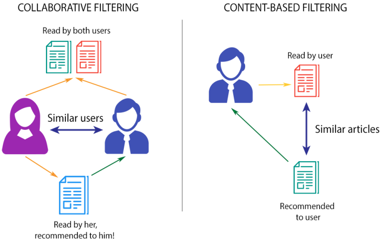


- Split data into overlapping windows of 10 seconds
- Fingerprint each window using the spectrogram (i.e., compute a binary string representing the reading in the window).
- All-pairs search for windows with high Jaccard similarity.



Online recommendation systems are often based on **collaborative filtering**. Simplest approach: find similar users and make recommendations based on those users.



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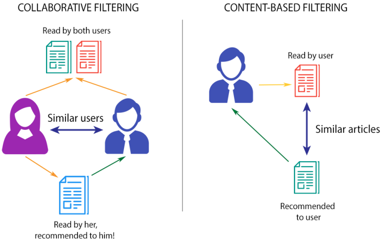

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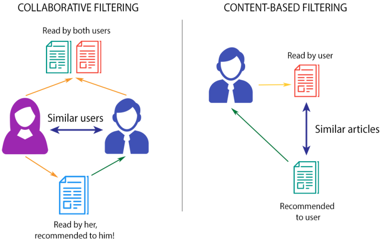

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- Netflix: look at sets of movies watched. Amazon: look at products purchased, etc.

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- **Fake Reviews**: Very common on websites like Amazon. Detection often looks for (near) duplicate reviews on similar products, which have been copied. 'Near duplicate' can be measured with shingles + Jaccard similarity.
- **Lateral phishing**: Phishing emails sent to addresses at a business coming from a legitimate email address at the same business that has been compromised.
  - One method of detection looks at the recipient list of an email and checks if it has small Jaccard similarity with any previous recipient lists. If not, the email is flagged as possible spam.

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- This is what we will cover next time. Using more random hashing!

Questions?