COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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University of Massachusetts Amherst. Spring 2020.

Lecture 3

By Thursday:

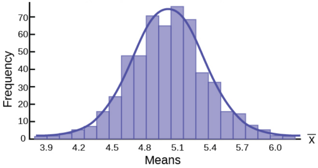
- Sign up for Piazza.
- Sign up for Gradescope (code on class website) and fill out the Gradescope consent poll on Piazza. Contact me via email if you don't consent to use Gradescope.
- First problem set will be **available in the next day or two, due 2/14.**

Last Class We Covered:

- Markov's inequality: the most fundamental concentration bound.
- Random hash functions, collision free hashing, and two-level hashing (analysis with linearity of expectation and Markov's inequality.)
- 2-universal and pairwise independent hash functions.
- Chebyshev's inequality and an elementary proof of the law of large numbers.

Today: We'll see even stronger concentration bounds than Chebyshev's inequality – exponential tail bounds.

- Will show a version of the central limit theorem.



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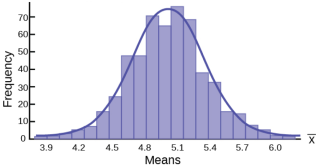
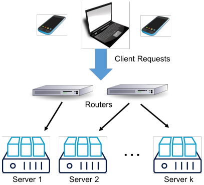


First: We'll show learn about the union bound and apply it to randomized load balancing.

Randomized Load Balancing:



- n requests randomly assigned to k servers using a random hash function.
- Letting $\mathbf{R}_i$ be the number of requests assigned to server i, $\mathbb{E}[\mathbf{R}_i] = \frac{n}{k}$ and we provision each server with the capacity to serve twice its expected load: $\frac{2n}{k}$ requests.
- What is the probability that a server exceeds its capacity?
- To apply Chebyshev's inequality, need to bound $\text{Var}[\mathbf{R}_i]$.

LOAD BALANCING VARIANCE

Recall that we can write R_i as:

$$R_i = \sum_{j=1}^{n} R_{i,j}$$

where $R_{i,j}$ is 1 if request j is assigned to server i and 0 o.w.

n: total number of requests, k: number of servers randomly assigned requests, R_i: number of requests assigned to server i.

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$$\begin{aligned}\text{Var}[\mathsf{R}_{i,j}] &= \mathbb{E}\left[\left(\mathsf{R}_{i,j} - \mathbb{E}[\mathsf{R}_{i,j}]\right)^2\right] \\ &= \Pr(\mathsf{R}_{i,j} = 1) \cdot \left(1 - \mathbb{E}[\mathsf{R}_{i,j}]\right)^2 + \Pr(\mathsf{R}_{i,j} = 0) \cdot \left(0 - \mathbb{E}[\mathsf{R}_{i,j}]\right)^2\end{aligned}$$

$$\frac{1}{k} \cdot \left(1 - \frac{1}{k}\right)^2 + \left(1 - \frac{1}{k}\right)\left(0 - \frac{1}{k}\right)^2$$

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\mathrm{Var}[\mathsf{R}_{i,j}] &= \mathbb{E}\left[\left(\mathsf{R}_{i,j} - \mathbb{E}[\mathsf{R}_{i,j}]\right)^2\right] \\
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&= \frac{1}{k} - \frac{1}{k^2} \leq \frac{1}{k}
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&= \frac{1}{k} - \frac{1}{k^2} \leq \frac{1}{k} \implies \text{Var}[\mathbf{R}_i] \leq \frac{n}{k}.
\end{aligned}$$

$\mathbb{E}\, R_i = \frac{n}{k}$

> n: total number of requests, k: number of servers randomly assigned requests, $\mathbf{R}_i$: number of requests assigned to server i.

BOUNDING THE LOAD VIA CHEBYSHEVS

Letting R_i be the number of requests sent to server i, $\mathbb{E}[R_i] = \frac{n}{k}$ and $\text{Var}[R_i] \leq \frac{n}{k}$.

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$$\Pr\left(R_i \geq \frac{2n}{k}\right) \leq \Pr\left(|R_i - \mathbb{E}[R_i]| \geq \frac{n}{k}\right)$$

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- Overload probability is extremely small when $k \ll n$!
- Might seem counterintuitive – bound gets worse as k grows.
- When k is large, the number of requests each server sees in expectation is very small so the law of large numbers doesn't 'kick in'.

n: total number of requests, k: number of servers randomly assigned requests, R_i: number of requests assigned to server i.

MAXIMUM SERVER LOAD

What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[R_i] = \frac{2n}{k}$. I.e., that some server is overloaded if we give each $\frac{2n}{k}$ capacity?

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$$\Pr\left(\max_i(\mathsf{R}_i) \geq \frac{2n}{k}\right) = \Pr\left(\left[\mathsf{R}_1 \geq \frac{2n}{k}\right] \cup \left[\mathsf{R}_2 \geq \frac{2n}{k}\right] \cup \ldots \cup \left[\mathsf{R}_k \geq \frac{2n}{k}\right]\right)$$

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How do we do this? Note that $R_1, \ldots, R_k$ are correlated in a somewhat complex way.

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Union Bound: For any random events $A_1, A_2, ..., A_k$,

$$\Pr\left(A_1 \cup A_2 \cup \ldots \cup A_k\right) \leq \Pr(A_1) + \Pr(A_2) + \ldots + Pr(A_k).$$

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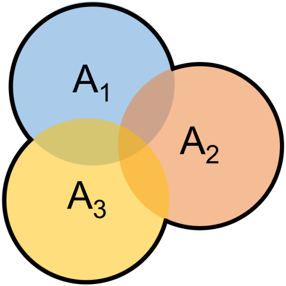

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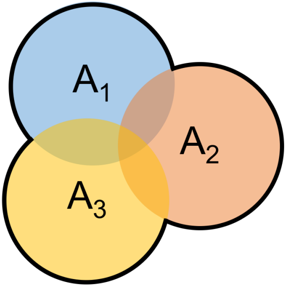

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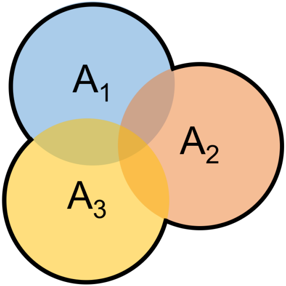

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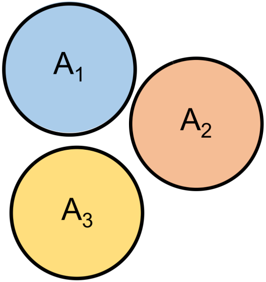

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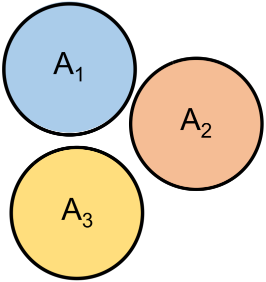

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On the first problem set, you will prove the union bound, as a consequence of Markov's inquality.

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$\frac{3n}{k}$

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As long as $k \leq O(\sqrt{n})$, with good probability, the maximum server load will be small (compared to the expected load).

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ANOTHER VIEW ON THIS PROBLEM

The number of servers must be small compared to the number of requests ($k = O(\sqrt{n})$) for the maximum load to be bounded in comparison to the expected load with good probability.

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- There are many requests routed to a relatively small number of servers so the load seen on each server is close to what is expected via law of large numbers.
- **A Useful Exercise:** Given n requests, and assuming all servers have fixed capacity C, how many servers should you provision so that with probability $\geq 99/100$ no server is assigned more than C requests?

> n: total number of requests, k: number of servers randomly assigned requests.

Questions on union bound, Chebyshev's inequality, random hashing?

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$$\mathbb{E}[\mathsf{H}] = \frac{n}{2} = 50 \text{ and } \mathrm{Var}[\mathsf{H}] = \frac{n}{4} = 25$$

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Markov's:

$\Pr(\mathsf{H} \geq 60) \leq .833$

$\Pr(\mathsf{H} \geq 70) \leq .714$

$\Pr(\mathsf{H} \geq 80) \leq .625$

FLIPPING COINS

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$$\mathbb{E}[\mathsf{H}] = \frac{n}{2} = 50 \text{ and } \mathrm{Var}[\mathsf{H}] = \frac{n}{4} = 25 \rightarrow s.d. = 5$$

Markov's:	Chebyshev's:	
$\Pr(\mathsf{H} \geq 60) \leq .833$	$\Pr(\mathsf{H} \geq 60) \leq .25$	$\frac{1}{2^2}$
$\Pr(\mathsf{H} \geq 70) \leq .714$	$\Pr(\mathsf{H} \geq 70) \leq .0625$	$1/4^2$
$\Pr(\mathsf{H} \geq 80) \leq .625$	$\Pr(\mathsf{H} \geq 80) \leq .0278$	$1/6^2$

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H has a simple Binomial distribution, so can compute these probabilities exactly.

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- What if we just apply Markov's inequality to even higher moments?

Consider any random variable X:

$$\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr\left((X - \mathbb{E}[X])^4 \geq t^4\right)$$

A FOURTH MOMENT BOUND

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$$\mathbb{E}\left[(H - \mathbb{E}[H])^4\right] = \mathbb{E}\left[\left(\sum_{i=1}^{100} H_i - 50\right)^4\right]$$

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- Apply Fourth Moment Bound: $\Pr\left(|H - \mathbb{E}[H]| \geq t\right) \leq \frac{1862.5}{t^4}$.

TIGHTER BOUNDS

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Chebyshev's:	4^{th} Moment:	In Reality:
$\Pr(\mathsf{H} \geq 60) \leq .25$	$\Pr(\mathsf{H} \geq 60) \leq .186$	$\Pr(\mathsf{H} \geq 60) = 0.0284$
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- Why monotonic? $\Pr(|\mathbf{X} - \mathbb{E}[\mathbf{X}]| > t) = \Pr(f(|\mathbf{X} - \mathbb{E}[\mathbf{X}]|) > f(t))$.

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- We will not cover the proofs in the this class.

Bernstein Inequality: Consider independent random variables $X_1, \ldots, X_n$ all falling in $[-M, M]$. Let $\mu = \mathbb{E}[\sum_{i=1}^n X_i]$ and $\sigma^2 = \text{Var}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \text{Var}[X_i]$. For any $t \geq 0$:

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- An exponentially stronger dependence on s!

Consider again bounding the number of heads **H** in $n = 100$ independent coin flips.

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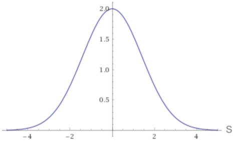

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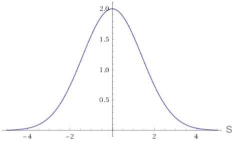

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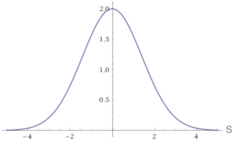

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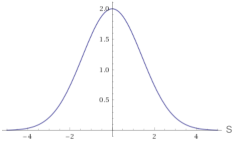

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Exercise: Using this can show that for $X \sim \mathcal{N}(0, \sigma^2)$: for any $s \geq 0$,

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$e^{-s^2/4}$

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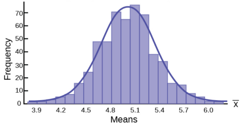
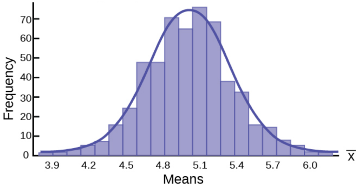
Central Limit Theorem Interpretation: Bernstein's inequality gives a quantitative version of the CLT. The distribution of the sum of *bounded* independent random variables can be upper bounded with a Gaussian (normal) distribution.



Stronger Central Limit Theorem: The distribution of the sum of *n bounded* independent random variables converges to a Gaussian (normal) distribution as *n* goes to infinity.



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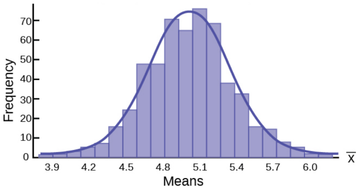

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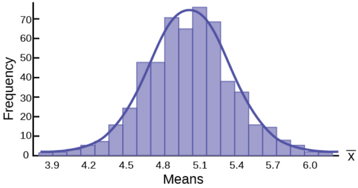

- Why is the Gaussian distribution is so important in statistics, science, ML, etc.?
- Many random variables can be approximated as the sum of a large number of small and roughly independent random effects. Thus, their distribution looks Gaussian by CLT.

A useful variation of the Bernstein inequality for binary (indicator) random variables is:

Chernoff Bound (simplified version): Consider independent random variables $X_1, \ldots, X_n$ taking values in $\{0, 1\}$. Let $\mu = \mathbb{E}[\sum_{i=1}^n X_i]$. For any $\delta \geq 0$

$$\Pr\left(\left|\sum_{i=1}^n X_i - \mu\right| \geq \delta\mu\right) \leq 2\exp\left(-\frac{\delta^2\mu}{2+\delta}\right).$$

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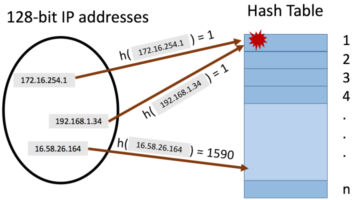
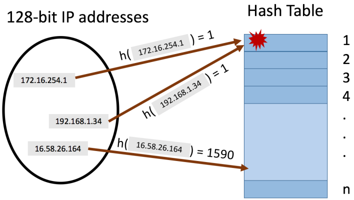
As δ gets larger and larger, the bound falls of exponentially fast.



We hash m values $x_1, \ldots, x_m$ using a random hash function into a table with $n = m$ entries.



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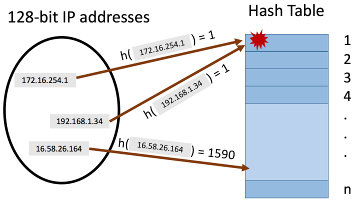
RETURN TO RANDOM HASHING



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What will be the maximum number of items hashed into the same location?

$O(n)$ $O(\log n)$ $O(\sqrt{n})$ $O(1/n)$

MAXIMUM LOAD IN RANDOMIZED HASHING

$m = n$

What will be the maximum number of items hashed into the same location? $O(\log m)$

m: total number of items hashed and size of hash table. $x_1, \ldots, x_m$: the items.
h: random hash function mapping $x_1, \ldots, x_m \to [m]$.

What will be the maximum number of items hashed into the same location? $O(\log m)$

Let $\mathbf{S}_i$ be the number of items hashed into position i and $\mathbf{S}_{i,j}$ be 1 if x_j is hashed into bucket i ($\mathbf{h}(x_j) = i$) and 0 otherwise.

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By the Chernoff Bound: for any $\delta \geq 0$,

$$\Pr(\mathbf{S}_i \geq 1 + \delta) \leq \Pr\left(\left|\sum_{i=1}^{n} \mathbf{S}_{i,j} - 1\right| \geq \delta\right) \leq 2\exp\left(-\frac{\delta^2}{2+\delta}\right)$$

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MAXIMUM LOAD IN RANDOMIZED HASHING

O(log m)

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$\Theta(\log m)$

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$$\leq \sum_{i=1}^{m} \Pr(\mathbf{S}_i \geq 20 \log m + 1) \leq m \cdot \frac{2}{m^{18}} = \frac{2}{m^{17}}.$$

$m \cdot \frac{1}{100m} = \frac{1}{100}$

$O(\log m)$

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- Using Chebyshev's inequality could only show the maximum load is bounded by $O(\sqrt{m})$ with good probability.
- The Chebyshev bound holds even with a pairwise independent hash function. The stronger Chernoff-based bound can be shown to hold with a k-wise independent hash function for $k = O(\log m)$.

Questions?

This concludes probability review/concentration bounds.