COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Cameron Musco

University of Massachusetts Amherst. Spring 2020.

Lecture 2

REMINDER

By Next Thursday 1/30:

- Sign up for Piazza.
- Sign up for Gradescope (code on class website) and fill out the Gradescope consent poll on Piazza. Contact me via email if you don't consent to use Gradescope.

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- Linearity of variance: $\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$ if X and Y are independent.
- Talked about an application of linearity to estimating the size of a CAPTCHA database.

Today:

- Finish up the CAPTCHA example and introduce Markov's inequality a fundamental concentration bound that let us prove that a random variable lies close to its expectation with good probability.
- Learn about random hash functions, which are a key tool in randomized methods for data processing. Probabilistic analysis via linearity of expectation.
- Start on Chebyshev's inequality: a concentration bound that is enough to prove a version of the law of large numbers.

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- In an attempt to verify this claim, you make m random security checks. If the database size is n then expected number of pairwise duplicate CAPTCHAS you see is:

$$\mathbb{E}[\mathbf{D}] = \sum_{i,j \in [m], i \neq j} \mathbb{E}[\mathbf{D}_{i,j}] = \frac{m(m-1)}{2n}.$$

If the database size is as claimed ($n = 1,000,000$) and you take $m = 1,000$ samples:

$$\mathbb{E}[\mathsf{D}] = \frac{m(m-1)}{2n} = .4995$$

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Concentration Inequalities: Bounds on the probability that a random variable deviates a certain distance from its mean.

- Useful in understanding how statistical tests perform, the behavior of randomized algorithms, the behavior of data drawn from different distributions, etc.

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The larger the deviation t, the smaller the probability.

Expected number of duplicate CAPTCHAS:

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You see $D = 10$ duplicates.

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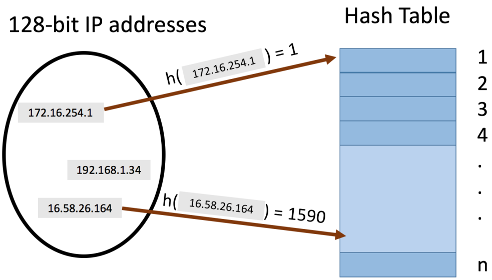
Classic Solution: Hash tables

- *Static hashing* since we won't worry about insertion and deletion today.



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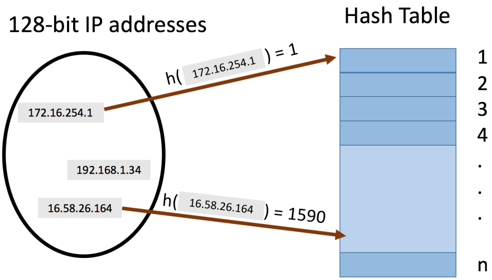

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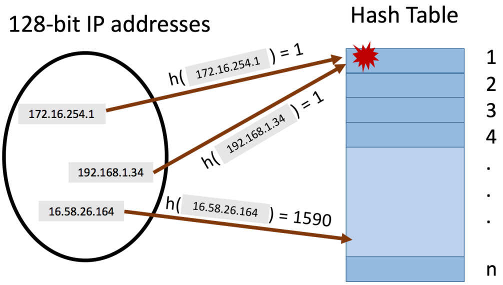

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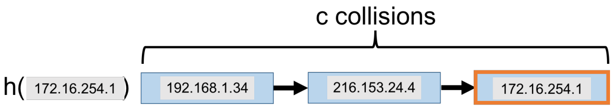
- Typically $|U| \gg n$. Many elements map to the same index.
- **Collisions:** when we insert m items into the hash table we may have to store multiple items in the same location (typically as a linked list).

Query runtime: $O(c)$ when the maximum number of collisions in a table entry is c (i.e., must traverse a linked list of size c).



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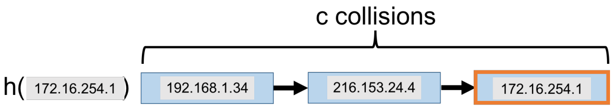

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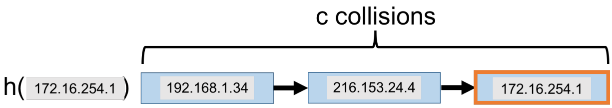

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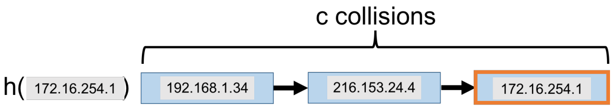

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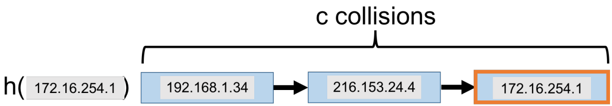

How Can We Bound *c*?

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Let $\mathsf{h} : U \to [n]$ be a random hash function.

- I.e., for $x \in U$, $\Pr(\mathsf{h}(x) = i) = \frac{1}{n}$ for all $i = 1, \ldots, n$ and $\mathsf{h}(x), \mathsf{h}(y)$ are independent for any two items $x \neq y$.

RANDOM HASH FUNCTION

$x_1 \ldots x_U$ $h(x_1) = i$ $h(x_2) = i$ $h(x_3) = i$

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- **Caveat 1:** It is *very expensive* to represent and compute such a random function. We will see how a hash function computable in $O(1)$ time function can be used instead.
- **Caveat 2:** In practice, often suffices to use hash functions like MD5, SHA-2, etc. that 'look random enough'.

$h(x)$

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Assuming we insert m elements into a hash table of size n, what is the expected total number of pairwise collisions?

$x_1 \quad x_2 \quad \ldots \quad x_m$

Let $C_{i,j} = 1$ if items i and j collide $(h(x_i) = h(x_j))$, and 0 otherwise. The number of pairwise ~~duplicates~~ collisions is:

$$C = \sum_{i,j \in [m], i \neq j} C_{i,j}.$$

x_i, x_j: pair of stored items, m: total number of stored items, n: hash table size, C: total pairwise collisions in table, h: random hash function.

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Identical to the CAPTCHA analysis from last class!

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COLLISION FREE HASHING

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$O(m)$

$$\Pr[\mathbf{C} = 0] = 1 - \Pr[\mathbf{C} \geq 1] \geq 1 - \frac{1}{8} = \frac{7}{8}.$$

Pretty good...but we are using $O(m^2)$ space to store m items...

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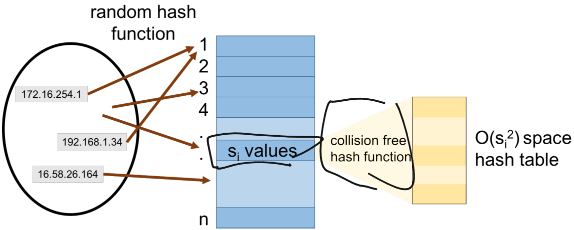

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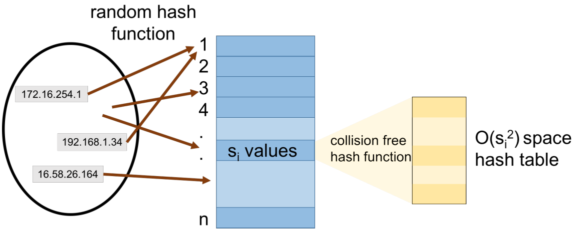

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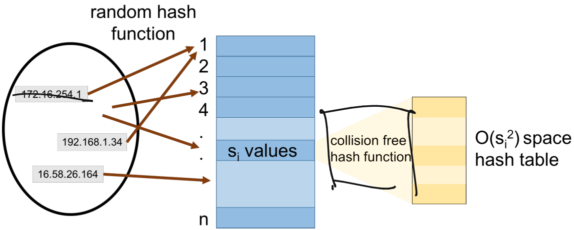

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- **Just Showed:** A random function is collision free with probability $\geq \frac{7}{8}$ so only requires checking $O(1)$ random functions in expectation to find a collision free one.

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x_j, x_k: stored items, n: hash table size, $\mathbf{h}$: random hash function, $\mathbf{S}$: space usage of two level hashing, $\mathbf{s}_i$: # items stored in hash table at position i.

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item in bucket i

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$$= \mathbb{E}\left[\sum_{j,k\in[m]} \mathbb{I}_{\mathbf{h}(x_j)=i} \cdot \mathbb{I}_{\mathbf{h}(x_k)=i}\right]$$

Collisions again!

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(1,1)
(1,2)
(2,1)
(1,3) ✓
(3,1) ✓

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O(m)

Near optimal space with $O(1)$ query time!

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Many Applications:

- Filter spam email addresses, phone numbers, suspect IPs, duplicate Tweets.
- Quickly check if an item has been stored in a cache or is new.
- Counting distinct elements (e.g., unique search queries.)

So Far: we have assumed a **fully random hash function** $\mathbf{h}(x)$ with $\Pr[\mathbf{h}(x) = i] = \frac{1}{n}$ for $i \in 1, \ldots, n$ and $\mathbf{h}(x), \mathbf{h}(y)$ independent for $x \neq y$.

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- To compute a random hash function we have to store a table of x values and their hash values. Would take at least $O(m)$ space and $O(m)$ query time if we hash m values. Making our whole quest for $O(1)$ query time pointless!

x	**h(x)**
x_1	45
x_2	1004
x_3	10
$\vdots$	$\vdots$
x_m	12

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Efficient Alternative: Let p be a prime with $p \geq |U|$. Choose random $\mathbf{a}, \mathbf{b} \in [p]$ with $\mathbf{a} \neq 0$. Let:

$$\mathbf{h}(x) = (\mathbf{a}x + \mathbf{b} \mod p) \mod n.$$

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Questions on linearity of expectation, Markov's, hashing?

NEXT STEP

1. We'll consider an application where our toolkit of linearity of expectation + Markov's inequality doesn't give much.

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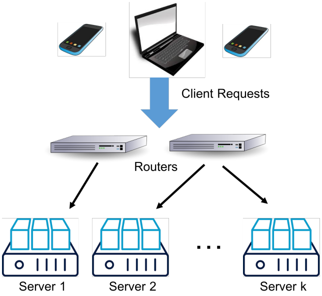
2. Then we'll show how a simple twist on Markov's can give a much stronger result.

Randomized Load Balancing:



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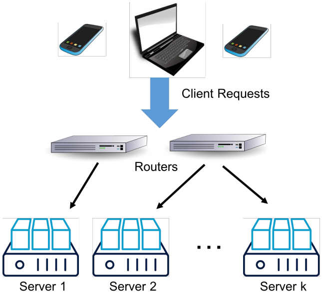

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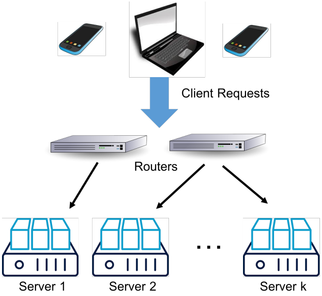

Simple Model: n requests randomly assigned to k servers. How many requests must each server handle?

- Often assignment is done via a random hash function. Why?

Expected Number of requests assigned to server i: $\frac{1}{k}$

$$\mathbb{E}[R_i] = \sum_{j=1}^{n} \mathbb{E}[\mathbb{I}_{\text{request } j \text{ assigned to } i}] = \sum_{j=1}^{n} \Pr[j \text{ assigned to } i] = \frac{n}{k}.$$

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WEAKNESS OF MARKOV'S

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Not great...half the servers may be overloaded.

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Chebyshev's inequality:

$$\Pr(|X - \mathbb{E}[X]| \geq t) \leq \frac{\text{Var}[X]}{t^2}.$$

(by plugging in the random variable $X - \mathbb{E}[X]$)

CHEBYSHEV'S INEQUALITY

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X: any random variable, t, s: any fixed numbers.

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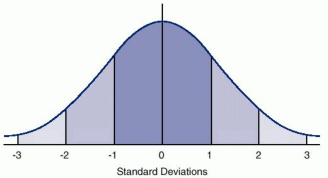
What is the probability that X falls s standard deviations from it's mean?



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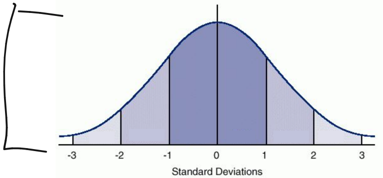

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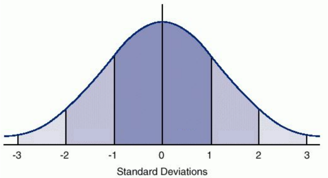

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Why is this so powerful?

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- Cannot show from vanilla Markov's inequality.

BACK TO LOAD BALANCING

Recall that R_i is the load on server i when n requests are randomly assigned to k servers.

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where $R_{i,j}$ is 1 if request j is assigned to server i and 0 o.w.

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where $\mathbf{R}_{i,j}$ is 1 if request j is assigned to server i and 0 o.w.

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\text{Var}[\mathbf{R}_{i,j}] &= \mathbb{E}\left[\left(\mathbf{R}_{i,j} - \mathbb{E}[\mathbf{R}_{i,j}]\right)^2\right] \\
&= \Pr(\mathbf{R}_{i,j} = 1) \cdot \left(1 - \mathbb{E}[\mathbf{R}_{i,j}]\right)^2 + \Pr(\mathbf{R}_{i,j} = 0) \cdot \left(0 - \mathbb{E}[\mathbf{R}_{i,j}]\right)^2 \\
&= \frac{1}{k} \cdot \left(1 - \frac{1}{k}\right)^2 + \left(1 - \frac{1}{k}\right) \cdot \left(0 - \frac{1}{k}\right)^2 \\
&= \frac{1}{k} - \frac{1}{k^2} \leq \frac{1}{k} \implies \text{Var}[\mathbf{R}_i] \leq \frac{n}{k}.
\end{aligned}$$

Applying Chebyshev's:

$$\Pr\left(\mathbf{R}_i \geq \frac{2n}{k}\right) \leq \Pr\left(|\mathbf{R}_i - \mathbb{E}[\mathbf{R}_i]| \geq \frac{n}{k}\right) \leq \frac{n/k}{n^2/k^2} = \frac{k}{n}.$$

Recall that $\mathbf{R}_i$ is the load on server i when n requests are randomly assigned to k servers.

$$\text{Var}[\mathbf{R}_i] = \sum_{j=1}^{n} \text{Var}[\mathbf{R}_{i,j}]$$

where $\mathbf{R}_{i,j}$ is 1 if request j is assigned to server i and 0 o.w.

$$\begin{aligned}
\text{Var}[\mathbf{R}_{i,j}] &= \mathbb{E}\left[\left(\mathbf{R}_{i,j} - \mathbb{E}[\mathbf{R}_{i,j}]\right)^2\right] \\
&= \Pr(\mathbf{R}_{i,j} = 1) \cdot \left(1 - \mathbb{E}[\mathbf{R}_{i,j}]\right)^2 + \Pr(\mathbf{R}_{i,j} = 0) \cdot \left(0 - \mathbb{E}[\mathbf{R}_{i,j}]\right)^2 \\
&= \frac{1}{k} \cdot \left(1 - \frac{1}{k}\right)^2 + \left(1 - \frac{1}{k}\right) \cdot \left(0 - \frac{1}{k}\right)^2 \\
&= \frac{1}{k} - \frac{1}{k^2} \leq \frac{1}{k} \implies \text{Var}[\mathbf{R}_i] \leq \frac{n}{k}.
\end{aligned}$$

Applying Chebyshev's:

$$\Pr\left(\mathbf{R}_i \geq \frac{2n}{k}\right) \leq \Pr\left(|\mathbf{R}_i - \mathbb{E}[\mathbf{R}_i]| \geq \frac{n}{k}\right) \leq \frac{n/k}{n^2/k^2} = \frac{k}{n}.$$

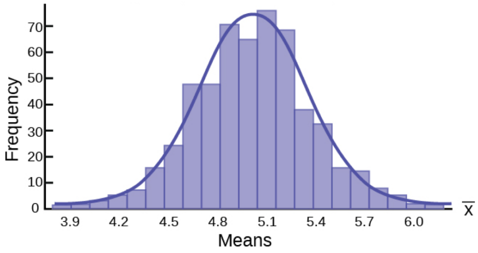
Overload probability is extremely small when $k \ll n$!

NEXT TIME

Chebyshev's Inequality: A quantitative version of the law of large numbers. The average of many independent random variables concentrates around its mean.

Chebyshev's Inequality: A quantitative version of the law of large numbers. The average of many independent random variables concentrates around its mean.

Chernoff Type Bounds: A quantitative version of the central limit theorem. The average of many independent random variables is distributed like a Gaussian.



Questions?