COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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University of Massachusetts Amherst. Spring 2020.
Lecture 17

LOGISTICS

- Problem Set 2 was released this weekend. Due Monday 4/13.
- See Piazza (and email from college) for clarification on P/F policy.

SUMMARY

Last Few Classes: Low-Rank Approximation and PCA

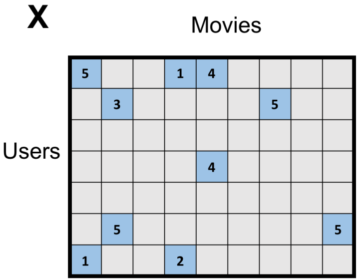
- Compress data that lies close to a k-dimensional subspace.
- Equivalent to finding a low-rank approximation of the data matrix $\mathbf{X}$: $\mathbf{X} \approx \mathbf{X}\mathbf{V}\mathbf{V}^T$ for orthonormal $\mathbf{V} \in \mathbb{R}^{d \times k}$.
- Optimal solution via PCA (eigendecomposition of $\mathbf{X}^T\mathbf{X}$ or equivalently, SVD of $\mathbf{X}$).
- Singular vectors of $\mathbf{X}$ are the eigenvectors of $\mathbf{X}\mathbf{X}^T$ and $\mathbf{X}^T\mathbf{X}$. Singular values squared are the eigenvalues.

This Class: Applications of low-rank approx. beyond compression.

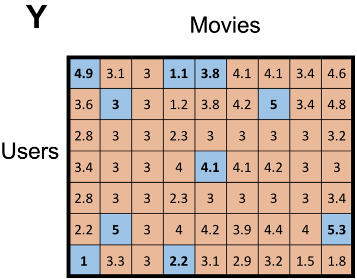
- Matrix completion and collaborative filtering
- Entity embeddings (word embeddings, node embeddings, etc.)
- Low-rank approximation for non-linear dimensionality reduction.
- Spectral graph theory, spectral clustering.

MATRIX COMPLETION

Consider a matrix $\mathbf{X} \in \mathbb{R}^{n\times d}$ which we cannot fully observe but believe is close to rank-k (i.e., well approximated by a rank k matrix). Classic example: the Netflix prize problem.



5			**1**	**4**				
	3					**5**		
				4				
	5							**5**
1			**2**					



4.9	3.1	3	**1.1**	**3.8**	4.1	4.1	3.4	4.6
3.6	**3**	3	1.2	3.8	4.2	**5**	3.4	4.8
2.8	3	3	2.3	3	3	3	3	3.2
3.4	3	3	4	**4.1**	4.1	4.2	3	3
2.8	3	3	2.3	3	3	3	3	3.4
2.2	**5**	3	4	4.2	3.9	4.4	4	**5.3**
1	3.3	3	**2.2**	3.1	2.9	3.2	1.5	1.8

Solve: $$Y = \underset{\text{rank}-k\ \mathbf{B}}{\arg\min} \sum_{\text{observed } (j,k)} \left[\mathbf{X}_{j,k} - \mathbf{B}_{j,k}\right]^2$$

Under certain assumptions, can show that $\mathbf{Y}$ well approximates $\mathbf{X}$ on both the observed and (most importantly) unobserved entries.

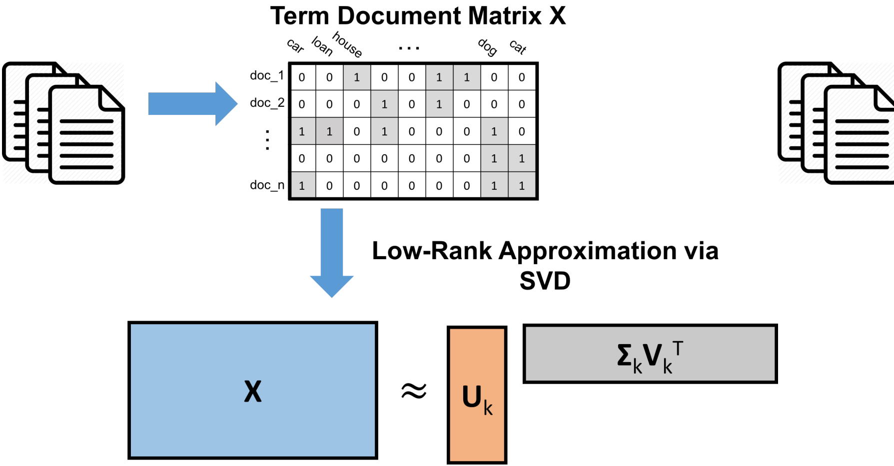
ENTITY EMBEDDINGS

Dimensionality reduction embeds d-dimensional vectors into d' dimensions. But what about when you want to embed objects other than vectors?

- Documents (for topic-based search and classification)
- Words (to identify synonyms, translations, etc.)
- Nodes in a social network

Usual Approach: Convert each item into a high-dimensional feature vector and then apply low-rank approximation.

EXAMPLE: LATENT SEMANTIC ANALYSIS

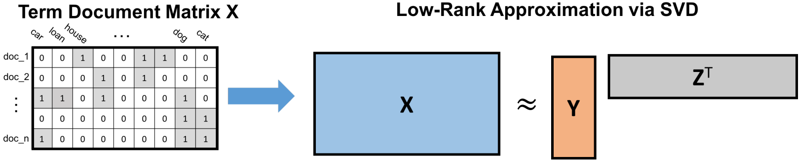


Term Document Matrix X

	car	loan	house		…			dog	cat
doc_1	0	0	1	0	0	1	1	0	0
doc_2	0	0	0	1	0	1	0	0	0
⋮	1	1	0	1	0	0	0	1	0
	0	0	0	0	0	0	0	1	1
doc_n	1	0	0	0	0	0	0	1	1

Low-Rank Approximation via SVD

EXAMPLE: LATENT SEMANTIC ANALYSIS



Term Document Matrix X

	car	loan	house		…			dog	cat
doc_1	0	0	1	0	0	1	1	0	0
doc_2	0	0	0	1	0	1	0	0	0
⋮	1	1	0	1	0	0	0	1	0
	0	0	0	0	0	0	0	1	1
doc_n	1	0	0	0	0	0	0	1	1

Low-Rank Approximation via SVD

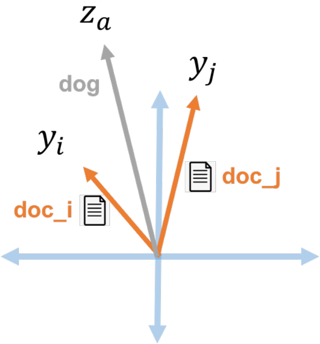
- If the error $\|\mathbf{X} - \mathbf{Y}\mathbf{Z}^T\|_F$ is small, then on average,

$$\mathbf{X}_{i,a} \approx (\mathbf{Y}\mathbf{Z}^T)_{i,a} = \langle \vec{y}_i, \vec{z}_a \rangle.$$

- I.e., $\langle \vec{y}_i, \vec{z}_a \rangle \approx 1$ when doc_i contains $word_a$.
- If doc_i and doc_j both contain $word_a$, $\langle \vec{y}_i, \vec{z}_a \rangle \approx \langle \vec{y}_j, \vec{z}_a \rangle \approx 1$.

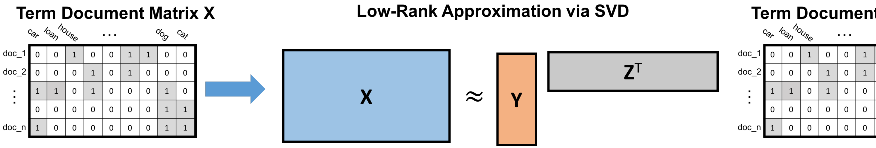
EXAMPLE: LATENT SEMANTIC ANALYSIS

If *doc*$_i$ and *doc*$_j$ both contain *word*$_a$, $\langle \vec{y}_i, \vec{z}_a \rangle \approx \langle \vec{y}_j, \vec{z}_a \rangle \approx 1$



Another View: Each column of **Y** represents a 'topic'. $\vec{y}_i(j)$ indicates how much *doc*$_i$ belongs to topic j. $\vec{z}_a(j)$ indicates how much *word*$_a$ associates with that topic.

EXAMPLE: LATENT SEMANTIC ANALYSIS



	car	loan	house	…				dog	cat
doc_1	0	0	1	0	0	1	1	0	0
doc_2	0	0	0	1	0	1	0	0	0
⋮	1	1	0	1	0	0	0	1	0
	0	0	0	0	0	0	0	1	1
doc_n	1	0	0	0	0	0	0	1	1

	car	loan	house	…		
doc_1	0	0	1	0	0	1
doc_2	0	0	0	1	0	1
⋮	1	1	0	1	0	0
	0	0	0	0	0	0
doc_n	1	0	0	0	0	0

- Just like with documents, $\vec{z}_a$ and $\vec{z}_b$ will tend to have high dot product if $word_a$ and $word_b$ appear in many of the same documents.
- In an SVD decomposition we set $\mathbf{Z} = \mathbf{\Sigma}_k \mathbf{V}_K^T$.
- The columns of $\mathbf{V}_k$ are equivalently: the top k eigenvectors of $\mathbf{X}^T\mathbf{X}$. The eigendecomposition of $\mathbf{X}^T\mathbf{X}$ is $\mathbf{X}^T\mathbf{X} = \mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^T$.
- What is the best rank-k approximation of $\mathbf{X}^T\mathbf{X}$? I.e. $\arg\min_{\text{rank}-k\ \mathbf{B}} \|\mathbf{X}^T\mathbf{X} - \mathbf{B}\|_F$
- $\mathbf{X}^T\mathbf{X} = \mathbf{V}_k\mathbf{\Sigma}_k^2\mathbf{V}_k^T = \mathbf{Z}\mathbf{Z}^T$.

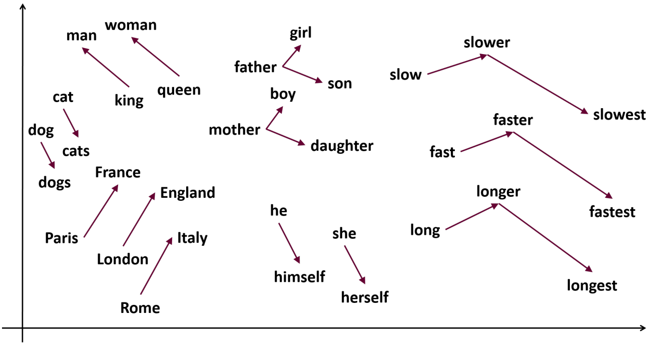
EXAMPLE: WORD EMBEDDING

LSA gives a way of embedding words into k-dimensional space.

- Embedding is via low-rank approximation of $\mathbf{X}^T\mathbf{X}$: where $(\mathbf{X}^T\mathbf{X})_{a,b}$ is the number of documents that both $word_a$ and $word_b$ appear in.

- Think about $\mathbf{X}^T\mathbf{X}$ as a similarity matrix (gram matrix, kernel matrix) with entry (a, b) being the similarity between $word_a$ and $word_b$.
- Many ways to measure similarity: number of sentences both occur in, number of times both appear in the same window of w words, in similar positions of documents in different languages, etc.
- Replacing $\mathbf{X}^T\mathbf{X}$ with these different metrics (sometimes appropriately transformed) leads to popular word embedding algorithms: word2vec, GloVe, fastText, etc.

EXAMPLE: WORD EMBEDDING

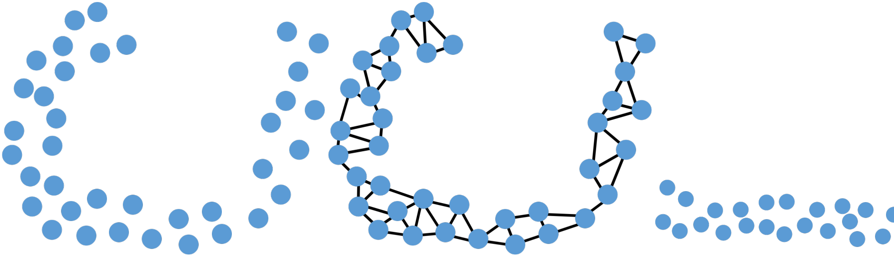


Note: word2vec is typically described as a neural-network method, but it is really just low-rank approximation of a specific similarity matrix. *Neural word embedding as implicit matrix factorization*, Levy and Goldberg.

SIMILARITY VIA GRAPHS

A common way of encoding similarity is via a graph. E.g., a *k*-nearest neighbor graph.

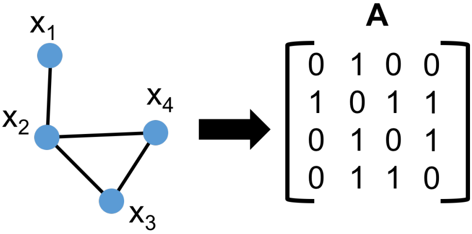
- Connect items to similar items, possibly with higher weight edges when they are more similar.



LINEAR ALGEBRAIC REPRESENTATION OF A GRAPH

Once we have connected n data points $x_1, \dots, x_n$ into a graph, we can represent that graph by its (weighted) adjacency matrix.

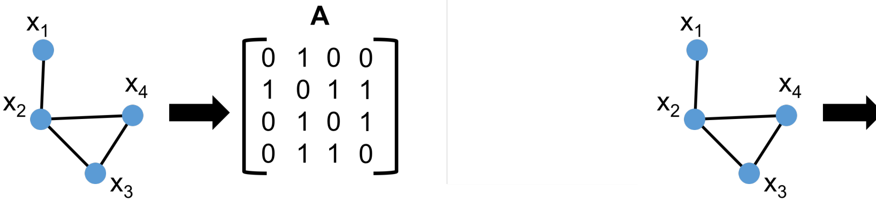
$$\mathbf{A} \in \mathbb{R}^{n \times n} \text{ with } \mathbf{A}_{i,j} = \text{ edge weight between nodes } i \text{ and } j$$



$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

In LSA example, when $\mathbf{X}$ is the term-document matrix, $\mathbf{X}^T\mathbf{X}$ is like an adjacency matrix, where $word_a$ and $word_b$ are connected if they appear in at least 1 document together (edge weight is $\#$ documents they appear in together).

NORMALIZED ADJACENCY MATRIX



$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

What is the sum of entries in the i^{th} column of A? The (weighted) degree of vertex i.

Often, $\mathbf{A}$ is normalized as $\bar{\mathbf{A}} = \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$ where $\mathbf{D}$ is the degree matrix.

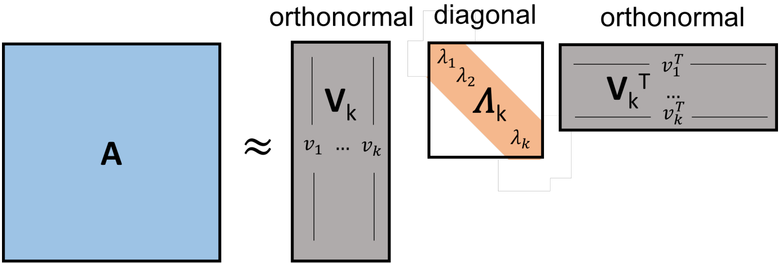
Spectral graph theory is the field of representing graphs as matrices and applying linear algebraic techniques.

ADJACENCY MATRIX EIGENVECTORS

How do we compute an optimal low-rank approximation of $\mathbf{A}$?

- Project onto the top k eigenvectors of $\mathbf{A}^T\mathbf{A} = \mathbf{A}^2$. These are just the eigenvectors of $\mathbf{A}$.

ADJACENCY MATRIX EIGENVECTORS



- Similar vertices (close with regards to graph proximity) should have similar embeddings. I.e., $\mathbf{V}_k(i)$ should be similar to $\mathbf{V}_k(j)$.

SPECTRAL EMBEDDING