COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Fall 2023.

Lecture 7

Summary

Last Class:

- Finish up exponential concentration bounds. Application to max load in hashing/load balancing. $O(\log n)$
- Bloom filters for storing a set with a small false positive rate.

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- Finish up exponential concentration bounds. Application to max load in hashing/load balancing.
- Bloom filters for storing a set with a small false positive rate.

This Class:

- Bloom Filter Analysis.
- Start on streaming algorithms
- The distinct items problem via random hashing.

Quiz

- Average time spent on homework: 18-20 hours.
- 18 people worked alone, 103 worked in groups. Mix of approaches to splitting up work in groups.

↳ splitting up problem set.

Quiz

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- 18 people worked alone, 103 worked in groups. Mix of approaches to splitting up work in groups.

$\mathbb{E}[X] = \sum_{i=1}^{n} \mu_i$

$\mathrm{Var}(X) = \sum_{i=1}^{n} \sigma_i$

$\mathbf{X}$ is the sum of independent random variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$, each with mean μ_i and variance σ_i. Each $\mathbf{X}_i$ takes on values in the range $[-5, 5]$. ~ M

Which of the following concentration bounds can you apply to show that $\mathbf{X}$ lies close to its expectation with good probability? Check all that apply.

Select one or more:

- a. ~~Markov's inequality.~~ — non-negative
- b. Chebyshev's inequality
- c. Bernstein's inequality.
- d. ~~Chernoff bound.~~ $\{0,1\}$

Check

Question 2

Not complete

Points out of 1.00

Flag question

Edit question

Bloom Filters

Chose k independent random hash functions $\mathbf{h}_1, \ldots, \mathbf{h}_k$ mapping the universe of elements $U \to [m]$.

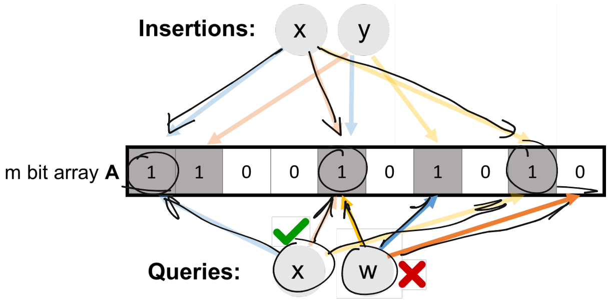
m items

- Maintain an array A containing m bits, all initially 0.
- *insert*(x): set all bits $A[\mathbf{h}_1(x)] = \ldots = A[\mathbf{h}_k(x)] := 1$.
- *query*(x): return 1 only if $A[\mathbf{h}_1(x)] = \ldots = A[\mathbf{h}_k(x)] = 1$.

$k = 3$

$m \cdot$ size of item space

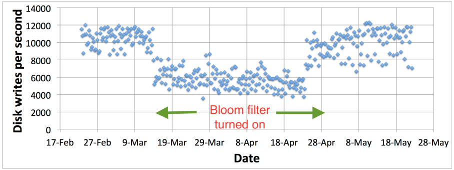
$O(m)$



No false negatives. False positives more likely with more insertions.

Applications: Caching

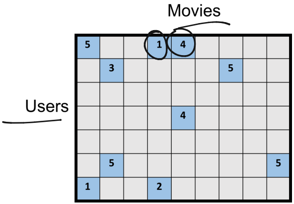
Akamai (Boston-based company serving 15 – 30% of all web traffic) applies bloom filters to prevent caching of 'one-hit-wonders' – pages only visited once fill over 75% of cache.



- When url x comes in, if $query(x) = 1$, cache the page at x. If not, run $insert(x)$ so that if it comes in again, it will be cached.
- **False positive:** A new url (possible one-hit-wonder) is cached. If the bloom filter has a false positive rate of $\delta = .05$, the number of cached one-hit-wonders will be reduced by at least 95%.

Applications: Databases

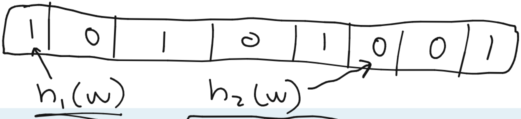
Distributed database systems, including Google Bigtable, Apache HBase, Apache Cassandra, and PostgreSQL use bloom filters to prevent expensive lookups of non-existent data.



- When a new rating is inserted for $(user_x, movie_y)$, add $(user_x, movie_y)$ to a bloom filter.
- Before reading $(user_x, movie_y)$ (possibly via an out of memory access), check the bloom filter, which is stored in memory.
- **False positive:** A read is made to a possibly empty cell. A $\delta = .05$ false positive rate gives a 95% reduction in these empty reads.

More Applications

- Database Joins: Quickly eliminate most keys in one column that don't correspond to keys in another.
- Recommendation systems: Bloom filters are used to prevent showing users the same recommendations twice.
- Spam/Fraud Detection:
  - Bit.ly and Google Chrome use bloom filters to quickly check if a url maps to a flagged site and prevent a user from following it.
  - Can be used to detect repeat clicks on the same ad from a single IP-address, which may be the result of fraud.
- Digital Currency: Some Bitcoin clients use bloom filters to quickly pare down the full transaction log to transactions involving bitcoin addresses that are relevant to them (SPV: simplified payment verification).



Consider a bloom filter where exactly 1/2 of the bits in the filter are set to 1, and the rest are set to 0. Consider running query(w) for some w that has not been inserted into the filter. If my implementation uses k independent, fully random hash functions, for k = 2, what is the probability at query(w) yields a false positive? Give your answer as an exact decimal number.

Answer:

Check

$$\Pr(FP) = \Pr\left(A[h_1(w)] = 1 \cap A[h_2(w)] = 1\right)$$

$$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

k hash functions

$$\frac{1}{2^k}$$

Analysis

For a bloom filter with m bits and k hash functions, the insertion and query time is $O(k)$.

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For a bloom filter with m bits and k hash functions, the insertion and query time is $O(k)$. How does the false positive rate δ depend on m, k, and the number of items inserted?

Step 1: What is the probability that after inserting n elements, the i^{th} bit of the array A is still 0?

$Pr(A[i] = 0)$ after inserting n elements

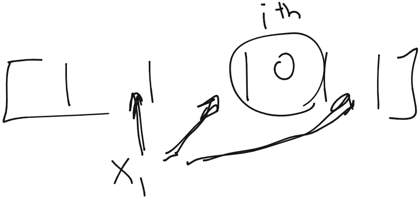
Analysis

For a bloom filter with m bits and k hash functions, the insertion and query time is $O(k)$. How does the false positive rate δ depend on m, k, and the number of items inserted?

Step 1: What is the probability that after inserting n elements, the i^{th} bit of the array A is still 0? $n \times k$ total hashes must not hit bit i.

$$\Pr(A[i] = 0) = \Pr\big(\mathbf{h}_1(x_1) \neq i \cap \ldots \cap \mathbf{h}_k(x_1) \neq i$$
$$\cap\, \mathbf{h}_1(x_2) \neq i \ldots \cap \mathbf{h}_k(x_2) \neq i \cap \ldots\big)$$

$n \cdot K$



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\begin{aligned}
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&\cap \mathbf{h}_1(x_2) \neq i \ldots \cap \mathbf{h}_k(x_2) \neq i \cap \ldots\big) \\
= &\underbrace{\Pr\big(\mathbf{h}_1(x_1) \neq i\big) \times \ldots \times \Pr\big(\mathbf{h}_k(x_1) \neq i\big) \times \Pr\big(\mathbf{h}_1(x_2) \neq i\big) \ldots}_{k \cdot n \text{ events each occuring with probability } \underline{1 - 1/m}}
\end{aligned}
$$

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\underline{\Pr(A[i] = 0)} &= \Pr\left(\mathbf{h}_1(x_1) \neq i \cap \ldots \cap \mathbf{h}_k(x_k) \neq i\right. \\
&\qquad\qquad \left.\cap\, \mathbf{h}_1(x_2) \neq i \ldots \cap \mathbf{h}_k(x_2) \neq i \cap \ldots\right) \\
&= \underbrace{\Pr\left(\mathbf{h}_1(x_1) \neq i\right) \times \ldots \times \Pr\left(\mathbf{h}_k(x_1) \neq i\right) \times \Pr\left(\mathbf{h}_1(x_2) \neq i\right) \ldots}_{k \cdot n \text{ events each occuring with probability } 1-1/m} \\
&= \underline{\left(1 - \frac{1}{m}\right)^{\underline{kn}}}
\end{aligned}
$$

Analysis

How does the false positive rate δ depend on m, k, and the number of items inserted?

Step 1: What is the probability that after inserting n elements, the i^{th} bit of the array A is still 0?

$$\Pr(A[i] = 0) = \left(1 - \frac{1}{m}\right)^{kn}$$

n: total number items in filter, m: number of bits in filter, k: number of random hash functions, $\mathbf{h}_1, \ldots \mathbf{h}_k$: hash functions, A: bit array, δ: false positive rate.

Analysis

How does the false positive rate δ depend on m, k, and the number of items inserted?

Step 1: What is the probability that after inserting n elements, the i^{th} bit of the array A is still 0?

$$\Pr(A[i] = 0) = \left(1 - \frac{1}{m}\right)^{kn} \approx e^{-\frac{kn}{m}}$$

$$\left(1 - \frac{1}{m}\right)^{m} \approx e^{-1}$$

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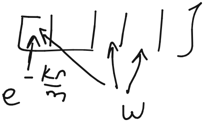
Analysis

How does the false positive rate δ depend on m, k, and the number of items inserted?

Step 1: What is the probability that after inserting n elements, the i^{th} bit of the array A is still 0?

$$\Pr(A[i]=0)=\left(1-\frac{1}{m}\right)^{kn}\approx e^{-\frac{kn}{m}}$$



Step 2: What is the probability that querying a new item w gives a false positive?

$$\left(1-e^{-\frac{kn}{m}}\right)^{k}$$

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Step 2: What is the probability that querying a new item w gives a false positive?

$$\Pr\left(A[\mathbf{h}_1(w)] = \ldots = A[\mathbf{h}_k(w)] = 1\right)$$
$$= \Pr(A[\mathbf{h}_1(w)] = 1) \times \ldots \times \Pr(A[\mathbf{h}_k(w)] = 1)$$

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Analysis

How does the false positive rate δ depend on m, k, and the number of items inserted?

Step 1: What is the probability that after inserting n elements, the i^{th} bit of the array A is still 0? $= \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{2}\cdot 1 = \frac{5}{8}$

$Pr(A[h_1 w]=1 \cap A[h_2(w)]=1)$

$n=1,\ k=2,\ m=2$

$$\Pr(A[i]=0) = \left(1-\frac{1}{m}\right)^{kn} \approx e^{-\frac{kn}{m}}$$

Step 2: What is the probability that querying a new item w gives a false positive? $A[h_1(w)]$ $A[h_2(w)]$

$h_1(w), h_2(w) \ldots h_k(w)$

$$\Pr\left(A[\mathbf{h}_1(w)] = \ldots = A[\mathbf{h}_k(w)] = 1\right)$$

$$= \Pr(A[\mathbf{h}_1(w)] = 1) \times \ldots \times \Pr(A[\mathbf{h}_k(w)] = 1)$$

$$= \boxed{\left(1-e^{-\frac{kn}{m}}\right)^k}$$

$A[h_1(w)]=1$

$A[h_2(w)]=1$

Actually Incorrect!

$Pr(A[h_1(w)]=1) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot 1$ $= 3/4$

$Pr(A[h_2(w)]=1) = 3/4$

$\frac{9}{16}$

n: total number items in filter, m: number of bits in filter, k: number of random hash functions, $\mathbf{h}_1, \ldots \mathbf{h}_k$: hash functions, A: bit array, δ: false positive rate.

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Actually Incorrect! Dependent events.

n: total number items in filter, m: number of bits in filter, k: number of random hash functions, $\mathbf{h}_1, \ldots \mathbf{h}_k$: hash functions, A: bit array, δ: false positive rate.

Correct Analysis Sketch

Step 1: To avoid dependence issues, condition on the event that the A has t zeros in it after n insertions, for some $t \leq m$. For a non-inserted element w, after conditioning on this event we correctly have:

$$\begin{aligned}\Pr(A[\mathbf{h}_1(w)] = \ldots = A[\mathbf{h}_k(w)] = 1) \\ = \Pr(A[\mathbf{h}_1(w)] = 1) \times \ldots \times \Pr(A[\mathbf{h}_k(w)] = 1).\end{aligned}$$

I.e., the events $A[\mathbf{h}_1(w)] = 1, \ldots, A[\mathbf{h}_k(w)] = 1$ are independent conditioned on the number of bits set in A. Why?

- Conditioned on this event, for any j, since $\mathbf{h}_j$ is a fully random hash function, $\Pr(A[\mathbf{h}_j(w)] = 1) = 1 - \frac{t}{m}$.
- Thus conditioned on this event, the false positive rate is $\left(1 - \frac{t}{m}\right)^k$.
- It remains to show that $\frac{t}{m} \approx e^{-\frac{kn}{m}}$ with high probability. We already have that $\mathbb{E}[\frac{t}{m}] = \frac{1}{m}\sum_{i=1}^{m} \Pr(A[i] = 0) \approx e^{-\frac{kn}{m}}$.

Correct Analysis Sketch

Need to show that the number of zeros t in A after n insertions is bounded by $O\left(e^{-\frac{kn}{m}}\right)$ with high probability.

Can apply Theorem 2 of:
`http://cglab.ca/~morin/publications/ds/bloom-submitted.pdf`

False Positive Rate

False Positive Rate: with m bits of storage, k hash functions, and n items inserted $\delta \approx \left(1 - e^{\frac{-kn}{m}}\right)^k$.

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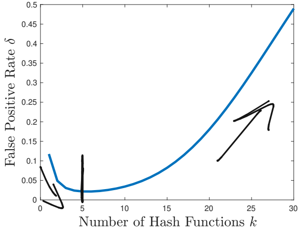
m, n
fix



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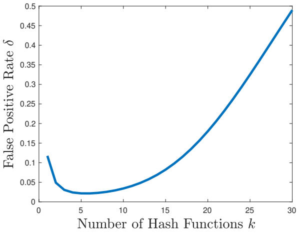

$(\ln 2) \cdot \frac{m}{n}$

- Can differentiate to show optimal number of hashes is $k = \ln 2 \cdot \frac{m}{n}$.
- Balances filling up the array vs. having enough hashes so that even when the array is pretty full, a new item is unlikely to yield a false positive.

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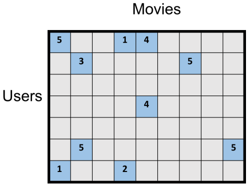

- Say we have 100 million users, each who have rated 10 movies.
- $n = 10^9 = n$ (user,movie) pairs with non-empty ratings.

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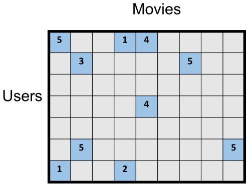

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- Allocate $m = 8n = 8 \times 10^9$ bits for a Bloom filter (1 GB).

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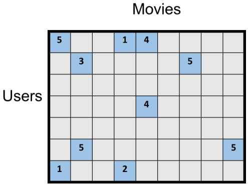

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- Set $k = \ln 2 \cdot \frac{m}{n} = 5.54 \approx 6$.

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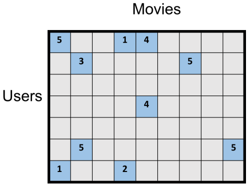
$\delta = .05$

k, m



$\frac{1}{2^k} = .05$

$k \approx 5$

$5 = \ln 2 \cdot \frac{m}{n}$

$n \cdot \frac{5}{\ln 2} = m$

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plug in $k = 6$

$\frac{1}{2}$

- False positive rate is $\approx \left(1 - e^{-k \cdot \frac{n}{m}}\right)^k \approx \frac{1}{2^k} \approx \frac{1}{2^{5.54}} = .021$.

$k = \ln 2 \cdot \frac{m}{n}$

$1 - e^{-k \cdot \frac{n}{m}} = 1 - e^{-\ln 2} = 1 - \frac{1}{2} = \frac{1}{2}$

Bloom Filter Note

An observation about Bloom filter space complexity:

$$\text{False Positive Rate: } \underline{\delta \approx \left(1 - e^{-\frac{kn}{m}}\right)}^{k}.$$

For an m-bit bloom filter holding n items, optimal number of hash functions k is: $\underline{k = \ln 2 \cdot \frac{m}{n}}$.

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Think Pair Share: If we want a false positive rate $< \frac{1}{2}$ how big does m need to be in comparison to n?

$$m = O(\log n), \; m = O(\sqrt{n}), \; \boxed{m = O(n)}, \; m = O(n^2)?$$

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$$m = O(\log n),\ m = O(\sqrt{n}),\ m = O(n),\ m = O(n^2)?$$

If $m = \frac{n}{\ln 2}$, optimal $k = 1$, and failure rate is:

$$\ln 2 \cdot \frac{m}{n} = 1$$

$$\delta = \left(1 - e^{-\frac{n/\ln 2}{n}}\right)^{1} = \left(1 - \frac{1}{2}\right)^{1} = \frac{1}{2}.$$

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$$m = O(\log n),\ m = O(\sqrt{n}),\ m = O(n),\ m = O(n^2)?$$

log(U)

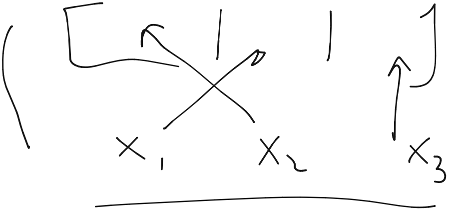
If $m = \frac{n}{\ln 2}$, optimal $k = 1$, and failure rate is:

$$\delta = \left(1 - e^{-\frac{n/\ln 2}{n}}\right)^1 = \left(1 - \frac{1}{2}\right)^1 = \frac{1}{2}.$$

$m = 8n$

$m = 5n$

I.e., storing n items in a bloom filter requires $O(n)$ space. So what's the point? Truly $O(n)$ bits, rather than $O(n \cdot \text{item size})$.



Questions on Bloom Filters?